A GENERALIZATION OF A LEIBNIZ GEOMETRICAL THEOREM

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Abstract:
In this article we present a generalization of a Leibniz's theorem in geometry and an application of this.

Leibniz's theorem. Let M be an arbitrary point in the plane of the triangle ABC, then $MA^2 + MB^2 + MC^2 = \frac{1}{3}(a^2 + b^2 + c^2) + 3MG^2$, where G is the centroid of the triangle. We generalize this theorem:

Theorem. Let's consider $A_1, A_2, ..., A_n$ arbitrary points in space and G the centroid of this points system; then for an arbitrary point M of the space is valid the following equation:

$$\sum_{i=1}^{n} MA_i^2 = \frac{1}{n} \sum_{1 \le i < j \le n} A_i A_j^2 + n \cdot MG^2 .$$

Proof. First, we interpret the centroid of the n points system in a recurrent way.

If $n = 2$ then is the midpoint of the segment.

If $n = 3$, then it is the centroid of the triangle.

Suppose that we found the centroid of the $n-1$ points created system. Now we join each of the n points with the centroid of the $n-1$ points created system; and we obtain n bisectors of the sides. It is easy to show that these n medians are concurrent segments. In this manner we obtain the centroid of the n points created system. We'll denote G_i the centroid of the A_k, $k = 1, 2, ..., i-1, i+1, ..., n$ points created system. It can be shown that $(n-1)A_i G = GG_i$. Now by induction we prove the theorem.

If $n = 2$ the $MA_1^2 + MA_2^2 = \frac{1}{2} A_1 A_2^2 + 2MG^2$

or

$$MG^2 = \frac{1}{4}\left(2\left(MA_1^2 + MA_2^2\right)\right),$$

where G is the midpoint of the segment A_1A_2. The above formula is the side bisector's formula in the triangle MA_1A_2. The proof can be done by Stewart's theorem, cosines theorem, generalized theorem of Pythagoras, or can be done vectorial. Suppose that the assertion of the theorem is true for $n = k$. If $A_1, A_2, ..., A_k$ are arbitrary points in space, G_0 is the centroid of this points system, then we have the following relation:

$$\sum_{i=1}^{k} MA_i^2 = \frac{1}{k} \sum_{1 \le i < j \le k} A_i A_j^2 + k \cdot MG_0^k.$$

Now we prove for $n = k + 1$.

Let $A_{k+1} \notin \{A_1, A_2, ..., A_k, G_0\}$ be an arbitrary point in the space and let G be the centroid of the $A_1, A_2, ..., A_k, A_{k+1}$ points system. Taking into account that G is on the segment $A_{k+1}G_0$ and $k \cdot A_{k+1}G = GG_0$, we apply Stewart's theorem to the points M, G_0, G, A_{k+1}, from where:

$$MA_{k+1}^2 \cdot GG_0 + MG_0^2 \cdot GA_{k+1} - MG^2 \cdot A_{k+1}G_0 = GG_0 \cdot GA_{k+1} \cdot A_{k+1}G_0.$$

According to the previous observation $A_{k+1}G = \frac{k}{k+1} A_{k+1}G_0$

and $GG_0 = \frac{k}{k+1} A_{k+1}G_0$.

Using these, the above relation becomes:

$$MA_{k+1}^2 + k \cdot MG_0^2 = \frac{k}{k+1} A_{k+1}G_0^2 + (k+1)MG^2.$$

From here

$$k \cdot MG_0^2 = \sum_{i=1}^{k} MA_i^2 - \frac{1}{k} \sum_{1 \le i < j \le k} A_i A_j^2.$$

From the supposition of the induction, with $M \equiv A_{k+1}$ as substitution, we obtain

$$\sum_{i=1}^{k} A_i A_j^2 = \frac{1}{k} \sum_{1 \le i < j \le k} A_i A_j^2 + k \cdot A_{k+1}G_0^2$$

and thus

$$\frac{k}{k+1} A_{k+1}G_0^2 = \frac{1}{k+1} \sum_{i=1}^{k} A_i A_{k+1}^2 - \frac{1}{k(k+1)} \sum_{1 \le i < j \le k} A_i A_j^2.$$

Substituting this in the above relation we obtain that

$$\sum_{i=1}^{k+1} MA_i^2 = \left(\frac{1}{k} - \frac{1}{k(k+1)} \right) \sum_{1 \le i < j \le k} A_i A_j^2 + \frac{1}{k+1} \sum_{i=1}^{k} A_i A_{k+1}^2 + (k+1)MG^2 =$$

$$= \frac{1}{k+1} \sum_{1 \le i < j \le k+1} A_i A_j^2 + (k+1)MG^2.$$

With this we proved that our assertion is true for $n = k + 1$. According to the induction, it is true for every $n \ge 2$ natural numbers.

Application 1. If the points $A_1, A_2, ..., A_n$ are on the sphere with the center O and radius R, then using in the theorem the substitution $M \equiv O$ we obtain the identity:

$$OG^2 = R^2 - \frac{1}{n^2} \sum_{1 \le i < j \le n} A_i A_j^2.$$

In case of a triangle: $OG^2 = R^2 - \frac{1}{9}\left(a^2 + b^2 + c^2\right)$.

In case of a tetrahedron: $OG^2 = R^2 - \frac{1}{16}\left(a^2 + b^2 + c^2 + d^2 + e^2 + f^2\right)$.

Application 2. If the points $A_1, A_2, ..., A_n$ are on the sphere with the center O and radius R, then $\sum_{1\le i<j\le n} A_iA_j^2 \le n^2R^2$.

The equality holds if and only if $G \equiv O$. In case of a triangle: $a^2 + b^2 + c^2 \le 9R^2$, in case of a tetrahedron: $a^2 + b^2 + c^2 + d^2 + e^2 + f^2 \le 16R^2$.

Application 3. Using the arithmetic and harmonic mean inequality, from the previous application, it results the following inequality:

$$\sum_{1\le i<j\le n} \frac{1}{A_iA_j^2} \ge \frac{(n-1)^2}{4R^2}.$$

In the case of a triangle: $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \ge \frac{1}{R^2}$, in case of a tetrahedron:

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{e^2} + \frac{1}{f^2} \ge \frac{9}{4R^2}.$$

Application 4. Considering the Cauchy-Buniakowski-Schwarz inequality from the Application 2, we obtain the following inequality:

$$\sum_{1\le i<j\le n} A_iA_j^2 \le nR\sqrt{\frac{n(n-1)}{2}}.$$

In case of a triangle: $a + b + c \le 3\sqrt{3}R$, in case of a tetrahedron:

$$a + b + c + d + e + f \le 4\sqrt{6}R.$$

Application 5. Using the arithmetic and harmonic mean inequality, from the previous application we obtain the following inequality

$$\sum_{1\le i<j\le n} \frac{1}{A_iA_j^2} \ge \frac{(n-1)\sqrt{n(n-1)}}{2R\sqrt{2}}.$$

In case of a triangle: $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ge \frac{\sqrt{3}}{R}$, in case of a tetrahedron:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{1}{f} \ge \frac{3}{R}\sqrt{\frac{3}{2}}.$$

Application 6. Considering application 3, we obtain the following inequality:

$$\frac{n^2(n-1)^2}{4} \le \left(\sum_{1\le i<j\le n} A_iA_j^k\right)\left(\sum_{1\le i<j\le n} \frac{1}{A_iA_j^k}\right) \le$$

$$\le \begin{cases} \dfrac{(M+m)^2 n^2 (n-1)^2}{16M\cdot m} & \text{if } \dfrac{n(n-1)}{2} \text{ is even,} \\ \dfrac{(M+m)^2 n^2 (n-1)^2 - 4(M-m)^2}{16M\cdot m} & \text{if } \dfrac{n(n-1)}{2} \text{ is odd} \end{cases}$$

where $m = \min\left\{A_i A_j^k\right\}$ and $M = \max\left\{A_i A_j^k\right\}$. In case of a triangle:

$$9 \le \left(a^k + b^k + c^k\right)\left(a^{-k} + b^{-k} + c^{-k}\right) \le \frac{2M^2 + 5M\cdot m + 2m^2}{M\cdot m},$$

in case of a tetrahedron:

$$36 \le \left(a^k + b^k + c^k + d^k + e^k + f^k\right)\left(a^{-k} + b^{-k} + c^{-k} + d^{-k} + e^{-k} + f^{-k}\right) \le \frac{9\left(M+m\right)^2}{M\cdot m}.$$

Application 7. Let $A_1, A_2, ..., A_n$ be the vertexes of the polygon inscribed in the sphere with the center O and radius R. First we interpret the orthocenter of the inscribable polygon $A_1A_2...A_n$. For three arbitrary vertexes, corresponds one orthocenter. Now we take four vertexes. In the obtained four orthocenters of the triangles we construct the circles with radius R, which have one common point. This will be the orthocenter of the inscribable quadrilateral. We continue in the same way. The circles with radius R that we construct in the orthocenters of the $n-1$ sides inscribable polygons have one common point. This will be the orthocenter of the n sides, inscribable polygon. It can be shown that O, H, G are collinear and $n\cdot OG = OH$. From the first application

$$OH^2 = n^2R^2 - \sum_{1\le i<j\le n} A_iA_j^2$$

and

$$GH^2 = \left(n-1\right)^2 R^2 - \left(1 - \frac{1}{n}\right)^2 \sum_{1\le i<j\le n} A_iA_j^2.$$

In case of a triangle $OH^2 = 9R^2 - \left(a^2 + b^2 + c^2\right)$ and $GH^2 = 4R^2 - \dfrac{4}{9}\left(a^2 + b^2 + c^2\right)$.

Application 8. In the case of an $A_1A_2...A_n$ inscribable polygon $\sum_{1\le i<j\le n} A_iA_j^2 = n^2R^2$ if and only if $O \equiv H \equiv G$. In case of a triangle this is equivalent with an equilateral triangle.

Application 9. Now we compute the length of the midpoints created by the $A_1, A_2, ..., A_n$ space points system. Let $S = \{1, 2, ..., i-1, i+1, ..., n\}$ and G_0 be the centroid of the A_k, $k \in S$, points system. By substituting $M \equiv A_i$ in the theorem, for the length of the midpoints we obtain the following relation:

$$A_iG_0^2 = \frac{1}{n-1}\sum_{k\in S} A_iA_k^2 - \frac{1}{\left(n-1\right)^2} \sum_{u,v\in S: u\ne v} A_uA_v^2.$$

Application 10. In case of a triangle $m_a^2 = \frac{2\left(b^2+c^2\right)-a^2}{4}$ and its permutations. From here:

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}\left(a^2+b^2+c^2\right),$$

$$m_a^2 + m_b^2 + m_c^2 \le \frac{27}{4}R^2,$$

$$m_a + m_b + m_c \le \frac{9}{2}R.$$

Application 11. In case of a tetrahedron $m_a^2 = \frac{1}{9}\left(3\left(a^2+b^2+c^2\right)-\left(d^2+e^2+f^2\right)\right)$ and its permutations.

From here:

$$\sum m_a^2 = \frac{4}{9}\left(\sum a^2\right),$$

$$\sum m_a^2 \le \frac{64}{9}R^2,$$

$$\sum m_a \le \frac{16}{3}R.$$

Application 12. Denote $m_{a,f}$ the length of the segments, which join midpoint of the a and f skew sides of the tetrahedron (bimedian). In the interpretation of the application $9m_{a,f}^2 = \frac{1}{4}\left(b^2+c^2+d^2+e^2-a^2-f^2\right)$ and its permutations.

From here

$$m_{a,f}^2 + m_{b,d}^2 + m_{c,e}^2 = \frac{1}{4}\left(\sum a^2\right),$$

$$m_{a,f}^2 + m_{b,d}^2 + m_{c,e}^2 \le 4R^2,$$

$$m_{a,f} + m_{b,d} + m_{c,e} \le 2R\sqrt{3}.$$

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