INEQUALITIES FOR THE INTEGER PART FUNCTION

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In this article we will prove some inequalities for the integer part function, and we’ll give some applications in the number theory.

Theorem 1. For any \( x, y > 0 \) we have the following inequality:
\[
[5x] + [5y] \geq [3x + y] + [3y + x],
\]
where \([ \ ]\) means the integer part function.

Proof: We will use the notations: \( x_i = [x], y_i = [y], u = \{x\}, v = \{y\}, x_i, y_i \in \mathbb{N} \) and \( u, v \in (0,1) \). We can write the inequality (1) as:
\[
x_i + y_i + [5u] + [5v] \geq [3u + v] + [3v + u].
\]
We distinguish the following cases:

a) Let \( u \geq v \). If \( u \leq 2v \), then \( 5v \geq 3v + u \) and \( [5v] \geq [3v + u] \), analogously \( 5u \geq 3u + v \) and \( [5u] \geq [3u + v] \), from where by addition we obtain (1).

b) If \( u > 2v \) and \( 5u = a + b \), \( 5v = c + d \), \( a, c \in \mathbb{N} \), \( 0 \leq b < 1 \), \( 0 \leq d < 1 \), then we have to prove the following inequality:
\[
a + c + x_i + y_i \geq \left[ \frac{3a + c + 3b + d}{5} \right] + \left[ \frac{3c + a + 3d + b}{5} \right].
\]
But, considering that \( 1 > u > 2v \), we obtain \( 5 > 5u > 10v \), from where, \( 5 > a + b > 2c + 2d \), thus \( a + b < 5 \) and \( a \leq 4 \).

If \( a < 2c \), then \( a \leq 2c - 1 \) and \( a + 1 - 2c \leq 0 \) thus \( a + b - 2c < 0 \); contradiction with \( a + b - 2c > 2d \), thus \( 4 \geq a, a \geq 2c \), and \( 3b + d < 4, 3d + b < 4 \).

From \( 4 \geq a \geq 2c \) we have the cases from the following table, and in each of the nine cases is verified the inequality (2).

<table>
<thead>
<tr>
<th>a</th>
<th>4 4 4 3 3 2 2 1 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>c</td>
<td>2 1 0 0 1 1 0 0 0</td>
</tr>
</tbody>
</table>

Application 1. For any \( m, n \in \mathbb{N} \), \( (5m)!(5n)! \) is divisible by \( m!n!(3m+n)!(3a+m)! \).

Proof: If \( p \) is a prime number, the power exponent of \( p \) in decomposition of \( m! \)

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Together with Mihály Bencze and Florin Popovici
is
\[ \left[ \frac{m}{p} \right] + \left[ \frac{m}{p^2} \right] + \ldots \]

It is sufficient to prove that
\[ \left[ \frac{5m}{r} \right] + \left[ \frac{5n}{r} \right] \geq \left[ \frac{m}{r} \right] + \left[ \frac{n}{r} \right] + \left[ \frac{3m+n}{r} \right] + \left[ \frac{3n+m}{r} \right] \]
for any \( r \in \mathbb{N}, r \geq 2 \).

If \( m = rm_1 + x, n = m_1 + y, \) where \( 0 \leq x < r, 0 \leq y < r, m, n \in \mathbb{Z}, \) it is sufficient to prove that
\[ \left[ \frac{5x}{r} \right] + \left[ \frac{5y}{r} \right] \geq \left[ \frac{3x+y}{r} \right] + \left[ \frac{3y+x}{r} \right], \]
but this inequality verifies the theorem 1.

**Remark.** If \( x,y > 0, \) then we have the inequality:
\[ 5x + 5y \geq x + y + 3x+y + 3y+x. \]

**Theorem 2.** (Szilárd András). If \( x, y, z \geq 0, \) then we have the inequality:
\[ 3x + 3y + 3z \geq x + y + z + x+y + y+z + z+x. \]

**Application 2.** For any \( a, b, c \in \mathbb{N}, \) \((3a)! \cdot (3b)! \cdot (3c)!\) is divisible by
\( a! \cdot b! \cdot (a+b)! \cdot (b+c)! \cdot (c+a)!\).

**Proof:** Let \( k_1, k_2, k_3 \) be the biggest power for which
\( p^{k_1} \mid 3a! \), \( p^{k_2} \mid 3b! \), \( p^{k_3} \mid 3c! \)
respectively, and \( r_i, i \in 1,2,3,4,5,6 \) the biggest power for which
\( p^{r_i} \mid a!, p^{r_i} \mid b!, p^{r_i} \mid c!, p^{r_i} \mid a+b \), \( p^{r_i} \mid b+c \), \( p^{r_i} \mid c+a \)
respectively, then
\[ k_1 + k_2 + k_3 = \left( \left[ \frac{3a}{p} \right] + \left[ \frac{3a}{p^2} \right] + \ldots \right) + \left( \left[ \frac{3b}{p} \right] + \left[ \frac{3b}{p^2} \right] + \ldots \right) + \left( \left[ \frac{3c}{p} \right] + \left[ \frac{3c}{p^2} \right] + \ldots \right) \]
and
\[ \sum_{i=1}^{6} r_i \left( \left[ \frac{a}{p} \right] + \left[ \frac{a}{p^2} \right] + \ldots \right) + \left( \left[ \frac{b}{p} \right] + \left[ \frac{b}{p^2} \right] + \ldots \right) + \left( \left[ \frac{c}{p} \right] + \left[ \frac{c}{p^2} \right] + \ldots \right) + \left( \left[ \frac{a+b}{p} \right] + \left[ \frac{a+b}{p^2} \right] + \ldots \right) + \left( \left[ \frac{b+c}{p} \right] + \left[ \frac{b+c}{p^2} \right] + \ldots \right) + \left( \left[ \frac{c+a}{p} \right] + \left[ \frac{c+a}{p^2} \right] + \ldots \right). \]

We have to prove that \( k_1 + k_2 + k_3 \geq \sum_{i=1}^{6} r_i \), but this inequality reduces to theorem 2.

**Theorem 3.** If \( x, y, z \geq 0, \) then we have the inequality:
Application 3. If \( a, b, c \in \mathbb{N} \), then \( a!b!c!(a+b+c)! \) is divisible by \((2a)! (2b)! (2c)!\).

Theorem 4. If \( x,y \geq 0 \) and \( n,k \in \mathbb{N} \) such that \( n \geq k \geq 0 \), then we have the inequality:
\[
x + ny \geq k \quad x + n - k \quad x + y.
\]

Application 4. If \( a, b, c \in \mathbb{N} \) and \( n \geq k \), then \( (na)! (nb)! \) is divisible by \( a^k b^k a + b + n - k \).

Theorem 5. If \( x_k \geq 0, \quad (k = 1,2,...,n) \), then we have the inequality:
\[
2 \sum_{k=1}^{n} x_k \geq 2 \sum_{k=1}^{n} x_k + x_1 + x_2 + x_2 + x_3 + ... + x_n + x_1.
\]

Application 5. If \( a_k \in \mathbb{N}, \quad (k = 1,2,...,n) \), then \( \prod_{k=1}^{n} 2a_k! \) is divisible by \( \prod_{k=1}^{n} (a_k!)^2 (a_1 + a_2) (a_2 + a_3)...(a_n + a_1) \).

Theorem 6. If \( x_k \geq 0, \quad (k = 1,2,...,n) \), then we have the inequality:
\[
m \sum_{k=1}^{n} 2x_k + n \sum_{p=1}^{m} 2x_p \geq m \sum_{k=1}^{n} x_k + n \sum_{p=1}^{m} x_p + \sum_{k=1}^{n} \sum_{p=1}^{m} x_k + x_p.
\]

Application 6. If \( a_k \in \mathbb{N}, \quad (k = 1,2,...,n) \), then \( \prod_{k=1}^{n} (2a_k)! \prod_{p=1}^{m} (2a_p)! \) is divisible by \( \prod_{k=1}^{n} a_k! \prod_{p=1}^{m} a_p! \prod_{k=1}^{n} \prod_{p=1}^{m} a_k + a_p! \).

Theorem 7. If \( x,y \geq 1 \), then we have the inequality:
\[
\left\lfloor \sqrt{x} \right\rfloor + \left\lfloor \sqrt{y} \right\rfloor + \left\lfloor \sqrt{x+y} \right\rfloor \geq \left\lfloor \sqrt{2x} \right\rfloor + \left\lfloor \sqrt{2y} \right\rfloor
\]

Proof: By the concavity of the square root function:
\[
\sqrt{x+y} = \sqrt{\frac{2x + 2y}{2}} \geq \frac{1}{2} \sqrt{2x} + \frac{1}{2} \sqrt{2y} \geq \left\lfloor \frac{1}{2} \sqrt{2x} \right\rfloor + \left\lfloor \frac{1}{2} \sqrt{2y} \right\rfloor,
\]
it follows that:
\[
\left\lfloor \sqrt{x+y} \right\rfloor \geq \left\lfloor \frac{1}{2} \sqrt{2x} \right\rfloor + \left\lfloor \frac{1}{2} \sqrt{2y} \right\rfloor.
\]
Therefore, it is sufficient to show that

\[
2x + 2y + 2z \leq x + y + z + x+y+z.
\]
\[
\left[\sqrt{x}\right] + \left[\frac{1}{2}\sqrt{2x}\right] \geq \left[\sqrt{2x}\right] \text{ for } x \geq 1.
\]

The identity \( \left[x\right] + \left[x + \frac{1}{2}\right] \) has a straightforward proof. We’ll use it to replace \( \left[\frac{1}{2}\sqrt{2x}\right] \) with
\[
\left[\sqrt{2x}\right] - \left[\frac{1}{2}\sqrt{2x} + \frac{1}{2}\right].
\]

This yields \( \left[\sqrt{x}\right] \geq \left[\frac{1}{2}\sqrt{2x} + \frac{1}{2}\right] \), for \( x \geq 1 \).

This last inequality followed by the fact that \( x \geq 4 \) implies
\[
2 - \sqrt{2} \cdot \sqrt{x} > 1 \text{ or } \left[\sqrt{x}\right] > \left[\frac{1}{2}\sqrt{2x} + \frac{1}{2}\right]
\]
and \( 1 \leq x < 4 \) implies
\[
\frac{1}{2}\sqrt{2x} + \frac{1}{2} < 2.
\]

**Application 7.** If \( a, b \in \mathbb{N} \), then \( a!b!\left[\sqrt{a^2 + b^2}\right] \) is divisible by \( \left[\frac{a}{2}\right]!\left[\frac{b}{2}\right]! \).

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