K-Divisibility and K-Strong Divisibility Sequences

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A sequence of rational integers $g$ is called a **divisibility sequence** if and only if

$$n \mid m \Rightarrow g(n) \mid g(m)$$

for all positive integers $n, m$. [See [3] and [4]].

Also, $g$ is called a **strong divisibility sequence** if and only if

$$(g(n), g(m)) = g((n, m))$$

for all positive integers $n, m$. [See [1], [2], [3], [4], and [5]].

Of course, it is easy to show that the results of the Smarandache function $S(n)$ is neither a divisibility nor a strong divisibility sequence, because $4 \mid 20$ but $S(4) = 4$ does not divide $5 = S(20)$, and $\left((S(4), S(20)) = (4, 5) = 1 \neq 4 = S(4) = S(4, 20)\right)$.

a) However, is there an infinite subsequence of integers $M = \{m_1, m_2, \ldots\}$ such that $S$ is a divisibility sequence on $M$?

b) If $\{p_1, p_2, \ldots\}$ is the set of prime numbers, the $S$ is not a strong divisibility sequence on $P$, because for $i \neq j$ we have

$$(S(p_i), S(p_j)) = (p_i, p_j) = 1 \neq 0 = S(1) = S(p_i, p_j).$$

And the same question can be asked about $P$ as it was asked in part a).

We introduce the following two notions, which are generalizations of a “divisibility sequence” and “strong divisibility sequence” respectively.

1) A $k$-divisibility sequence, where $k \geq 1$ is an integer, is defined in the following way:

If

$$n \mid m \Rightarrow g(n) \mid g(m) \Rightarrow g(g(n)) \mid g\left(\underbrace{g(g(n)) \mid g\left(\cdots g(n) \mid g\left(\cdots g(m) \mid g\left(\cdots\right)_{k\text{ times}}\right)_{k\text{ times}}\right)}_{k\text{ times}}\right)$$

for all positive integers $n, m$.

For example, $g(n) = n!$ is a $k$-divisibility sequence.

Also, any constant sequence is a $k$-divisibility sequence.

2) A $k$-strong divisibility sequence, where $k \geq 1$ is an integer, is defined in the following way:

If $(g(n_1), g(n_2), \ldots, g(n_k)) = g((n_1, n_2, \ldots, n_k))$ for all positive integers $n_1, n_2, \ldots, n_k$. 

For example, \( g(n) = 2n \) is a k-strong divisibility sequence, because
\[
(2n_1, 2n_2, ..., 2n_k) = 2^k (n_1, n_2, ..., n_k) = g((n_1, n_2, ..., n_k)).
\]

**Remarks:** If \( g \) is a divisibility sequence and we apply its definition \( k \)-times, we obtain that \( g \) is a \( k \)-divisibility sequence for any \( k \geq 1 \). The converse is also true. If \( g \) is \( k \)-strong divisibility sequence, \( k \geq 2 \), then \( g \) is a strong divisibility sequence. This can be seen by taking the definition of a \( k \)-strong divisibility sequence and replacing \( n \) by \( n_1 \) and all \( n_2, ..., n_k \) by \( m \) to obtain
\[
(g(n), g(m), ..., g(m)) = g((n, m, ..., m)) \text{ or } (g(n), g(m)) = g((n, m)).
\]
The converse is also true, as
\[
(n_1, n_2, ..., n_k) = ((n_1, n_2), n_3, ..., n_k).
\]
Therefore, we found that:

a) The divisibility sequence notion is equivalent to a \( k \)-divisibility sequence, or a generalization of a notion is equivalent to itself.

Is there any paradox or dilemma?

b) The strong divisibility sequence is equivalent to the \( k \)-strong divisibility sequence notion

As before, a generalization of a notion is equivalent to itself.

Again, is there any paradox or dilemma?

**REFERENCES**


