A METHOD OF RESOLVING IN INTEGER NUMBERS OF CERTAIN NONLINEAR EQUATIONS

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Let’s consider a polynomial with integer coefficients, of degree $m$

$$P(X_1,..., X_n) = \sum_{0 \leq i_1 + \cdots + i_n \leq m} a_{i_1...i_n} X_1^{i_1} \cdots X_n^{i_n}$$

which can be decomposed in linear factors (which can eventually be established through the undetermined coefficients method):

$$P(X_1,..., X_n) = (A_1^{(1)} X_1 + \cdots + A_n^{(1)} X_n + A_{n+1}^{(1)}) \cdots (A_1^{(m)} X_1 + \cdots + A_n^{(m)} X_n + A_{n+1}^{(m)}) + B$$

with all $A_j^{(k)}$, $B$ in $\Theta$, but which by bringing to the same common denominator and by eliminating it from the equation $P(X_1,..., X_n) = 0$ they can be considered integers. Thus the equation transforms in the following system:

$$\begin{align*}
A_1^{(1)} X_1 + \cdots + A_n^{(1)} X_n + A_{n+1}^{(1)} &= D_1 \\
A_1^{(m)} X_1 + \cdots + A_n^{(m)} X_n + A_{n+1}^{(m)} &= D_m
\end{align*}$$

where $D_1,..., D_m$ are the divisors for $B$ and $D_1 \cdots D_m = B$.

We resolve separately each linear Diophantine equation and then we intersect the equations.

**Example.** Resolve in integer numbers the equation:

$$-2x^3 + 5x^2y + 4xy^2 - 3y^3 - 3 = 0.$$ 

We’ll write the equation in another format

$$(x + y)(2x - y)(-x + 3y) = 3.$$ 

Let $m$, $n$ and $p$ be the divisors of $3, m \cdot n \cdot p = 3$. Thus

$$\begin{align*}
x + y &= m \\
2x - y &= n \\
-x + 3y &= p
\end{align*}$$

For this system to be compatible it is necessary that

$$\begin{pmatrix}
1 & 1 & m \\
2 & -1 & n \\
-1 & 3 & p
\end{pmatrix} = 0,$$
or

\[ 5m - 4n - 3p = 0 \]  \hspace{1cm} (1)

In this case

\[ x = \frac{m + n}{3} \quad \text{and} \quad y = \frac{2m - n}{3} \]  \hspace{1cm} (2)

Because \( m, n, p \in \mathbb{Z} \), from (1) it results – by resolving in integer numbers – that:

\[
\begin{align*}
& m = 3k_1 - k_2 \\
& n = k_2 \\
& p = 5k_1 - 3k_2
\end{align*}
\]

which substituted in (2) will give us \( x = k_1 \) and \( y = 2k_1 - k_2 \). But \( k_2 \in D(3) = \{\pm 1, \pm 3\} \); thus the only solution is obtained for \( k_2 = 1, k_1 = 0 \) from where \( x = 0 \) and \( y = -1 \).

Analogue it can be shown that, for example the equation:

\[-2x^3 + 5x^2y + 4xy^2 - 3y^3 = 6\]

does not have solutions in integer numbers.

REFERENCES
