Several Metrical Relations Regarding the Anti-Bisector, the Anti-Symmedian, the Anti-Height and their Isogonal

Professor Ion Pătraşcu – Fraţii Buzeşti National College, Craiova 
Professor Florentin Smarandache – University of New Mexico, U.S.A.

We suppose known the definitions of the isogonal cevian and isometric cevian; we remind that the anti-bisector, the anti-symmedian, and the anti-height are the isometrics of the bisector, of the symmedian and of the height in a triangle.

It is also known the following Steiner (1828) relation for the isogonal cevians $AA_i$ and $AA'_i$:

$$\frac{BA_i}{CA_i} \cdot \frac{BA'_i}{CA'_i} = \left( \frac{AB}{AC} \right)^2$$

We’ll prove now that there is a similar relation for the isometric cevians

**Proposition**

In the triangle $ABC$ let consider $AA_i$ and $AA'_i$ two isometric cevians, then there exists the following relation:

$$\frac{\sin(BAA_i)}{\sin(CAA_i)} \cdot \frac{\sin(BAA'_i)}{\sin(CAA'_i)} = \left( \frac{\sin B}{\sin C} \right)^2 \quad (*)$$

**Proof**

The sinus theorem applied in the triangles $ABA_i, ACA_i$ implies (see above figure)

$$\frac{\sin(BAA_i)}{BA_i} = \frac{\sin B}{AA_i} \quad (1)$$

$$\frac{\sin(CAA_i)}{CA_i} = \frac{\sin C}{AA_i} \quad (2)$$

![Fig. 1](image-url)
From the relations (1) and (2) we retain
\[
\frac{\sin(BA_i)}{\sin(CA_i)} = \frac{\sin B \cdot BA_i}{\sin C \cdot CA_i}
\]
(3)

The sinus theorem applied in the triangles \(ACA_i, ABA_i\) leads to
\[
\frac{\sin(CA_i)}{AA_i} = \frac{\sin C}{AC}
\]
(4)
\[
\frac{\sin(BA_i)}{BA_i} = \frac{\sin B}{AA_i}
\]
(5)

From the relations (4) and (5) we obtain:
\[
\frac{\sin(BA_i)}{\sin(CA_i)} = \frac{\sin B \cdot BA_i}{\sin C \cdot CA_i}
\]
(6)

Because \(BA_i = CA_i\) and \(AC = BA_i\) (the cevians being isometric), from the relations (3) and (6) we obtain relation (*) from the proposition’s enunciation.

**Applications**

1. If \(AA_i\) is the bisector in the triangle \(ABC\) and \(AA_i\) is its isometric, that is an anti-bisector, then from (*) we obtain
\[
\frac{\sin(BA_i)}{\sin(CA_i)} = \left(\frac{\sin B}{\sin C}\right)^2
\]
(7)

Taking into account of the sinus theorem in the triangle \(ABC\) we obtain
\[
\frac{\sin(BA_i)}{\sin(CA_i)} = \left(\frac{AC}{AB}\right)^2
\]
(8)

2. If \(AA_i\) is symmedian and \(AA_i\) is an anti-symmedian, from (*) we obtain
\[
\frac{\sin(BA_i)}{\sin(CA_i)} = \left(\frac{AC}{AB}\right)^3
\]

Indeed, \(AA_i\) being symmedian it is the isogonal of the median \(AM\) and
\[
\frac{\sin(MB)}{\sin(MC)} = \frac{\sin B}{\sin C} \quad \text{and} \quad \frac{\sin(BA_i)}{\sin(CA_i)} = \frac{\sin(MC)}{\sin(MB)} = \frac{\sin C}{\sin B} = \frac{AB}{AC}
\]
3. If $A_A$ is a height in the triangle $ABC$, $A_i \in (BC)$ and $A'A_i$ is its isometric (anti-height), the relation (*) becomes.

$$\frac{\sin(BAA')}{\sin(CAA')} = \left(\frac{AC}{AB}\right)^2 \cdot \frac{\cos C}{\cos B}$$

Indeed

$$\sin(BAA') = \frac{BA_i}{AB} \cdot \sin(CAA') = \frac{CA_i}{AC}$$

therefore

$$\frac{\sin(BAA_i)}{\sin(CAA_i)} = \frac{AC \cdot BA_i}{AB \cdot CA_i}$$

From (*) it results

$$\frac{\sin(BAA_i)}{\sin(CAA_i)} = \frac{AC}{AB} \cdot \frac{CA_i}{BA_i}$$

or

$$CA_i = AC \cdot \cos C \quad \text{and} \quad BA_i = AB \cdot \cos B$$

therefore

$$\frac{\sin(BAA_i)}{\sin(CAA_i)} = \left(\frac{AC}{AB}\right)^2 \cdot \frac{\cos C}{\cos B}$$

4. If $A_A''$ is the isogonal of the anti-bisector $A'A_i$ then

$$\frac{BA''}{A''C} = \left(\frac{AB}{AC}\right)^3$$ (Maurice D’Ocagne, 1883)

**Proof**

The Steiner’s relation for $A'A_i$ and $A'A_i''$ is

$$\frac{BA_i}{A_iC} \cdot \frac{BA_i''}{A_i''C} = \left(\frac{AB}{AC}\right)^2$$

But $A_A$ is the bisector and according to the bisector theorem

$$\frac{BA_i}{CA_i} = \frac{AB}{AC}$$

but $BA_i' = CA_i$ and

$A_i' C = BA_i$ therefore

$$\frac{CA_i}{BA_i} = \frac{AB}{AC}$$

and we obtain the D’Ocagne relation
5. If in the triangle $ABC$ the cevian $AA_i''$ is isogonal to the symmedian $AA_i'$ then

$$\frac{BA_i''}{A_i'C} = \left(\frac{AB}{AC}\right)^4$$

**Proof**
Because $AA_i$ is a symmedian, from the Steiner’s relation we deduce that

$$\frac{BA_i}{CA_i} = \left(\frac{AB}{AC}\right)^2$$

The Steiner’s relation for $AA_i''$, $AA_i'$ gives us

$$\frac{BA_i'' \cdot BA_i'}{A_i'C \cdot CA_i'} = \left(\frac{AB}{AC}\right)^2$$

Taking into account the precedent relation, we obtain

$$\frac{BA_i''}{A_i'C} = \left(\frac{AB}{AC}\right)^4$$

6. If $AA_i'$ is the isogonal of the anti-height $AA_i'$ in the triangle $ABC$ in which the height $AA_i$ has $A_i \in (BC)$ then

$$\frac{BA_i'}{A_i'C} = \left(\frac{AB}{AC}\right)^3 \cdot \frac{\cos B}{\cos C}$$

**Proof**
If $AA_i$ is height in triangle $ABC$ $A_i \in (BC)$ then

$$\frac{BA_i}{A_i'C} = \frac{AB}{AC} \cdot \frac{\cos B}{\cos C}$$

Because $AA_i'$ is anti-median, we have $BA_i = CA_i'$ and $AC = BA_i'$ then

$$\frac{BA_i''}{A_i'C} = \frac{AC}{AB} \cdot \frac{\cos C}{\cos B}$$

**Observation**
The precedent results can be generalized for the anti-cevians of rang $k$ and for their isogonal.