Mixt-Linear Circles Adjointly Ex-Inscribed Associated to a Triangle

Ion Pătrașcu, Professor, The Frații Buzești College, Craiova, Romania
Florentin Smarandache, Professor, The University of New Mexico, U.S.A.

Abstract
In [1] we introduced the mixt-linear circles adjointly inscribed associated to a triangle, with emphasizes on some of their properties. Also, we’ve mentioned about mixt-linear circles adjointly ex-inscribed associated to a triangle.

In this article we’ll show several basic properties of the mixt-linear circles adjointly ex-inscribed associate to a triangle.

Definition 1
We define a mixt-linear circle adjointly ex-inscribed associated to a triangle, the circle tangent exterior to the circle circumscribed to a triangle in one of the vertexes of the triangle, and tangent to the opposite side of the vertex of that triangle.

Observation
In Fig.1 we constructed the mixt-linear circle adjointly ex-inscribed to triangle \(ABC\), which is tangent in \(A\) to the circumscribed circle of triangle \(ABC\), and tangent to the side \(BC\). Will call this the \(A\)-mixt-linear circle adjointly ex-inscribed to triangle \(ABC\). We note \(L_A\) the center of this circle.

**Remark**
In general, for a triangle exists three mixt-linear circles adjointly ex-inscribed. If the triangle \(ABC\) is isosceles with the base \(BC\), then we cannot talk about mixt-linear circles adjointly ex-inscribed associated to the isosceles triangle.

**Proposition 1**
The tangency point with the side \(BC\) of the \(A\)-mixt-linear circle adjointly ex-inscribed associated to the triangle is the leg of the of the external bisectrix of the angle \(BAC\).

**Proof**
Let \(D'\) the contact point with the side \(BC\) of the \(A\)-mixt-linear circle adjointly ex-inscribed and let \(A'\) the intersection of the tangent in the point \(A\) to the circumscribed circle to the triangle \(ABC\) with \(BC\) (see Fig. 1).

We have
\[
m\left(\angle AA'B\right)=\frac{1}{2}\left[m\left(\hat{B}\right)-m\left(\hat{C}\right)\right],
\]
(we supposed that \(m\left(\hat{B}\right)>m\left(\hat{C}\right)\)). The tangents \(AA', A'D'\) to the \(A\)-mixt-linear circle adjointly ex-inscribed are equal, therefore
\[
m\left(\angle D'AA'\right)=\frac{1}{4}m\left(\hat{B}-\hat{C}\right).
\]
Because
\[
m\left(\angle A' AB\right)=\frac{1}{2}m\left(\hat{C}\right)
\]
we obtain that
\[
m\left(\angle D' AB\right)=\frac{1}{2}\left[m\left(\hat{B}\right)+m\left(\hat{C}\right)\right]
\]
This relation shows that \(D'\) is the leg of the external bisectrix of the angle \(BAC\).

**Proposition 2**
The \(A\)-mixt-linear circle adjointly ex-inscribed to triangle \(ABC\) intersects the sides \(AB, AC\), respectively, in two points of a cord which is parallel to \(BC\).

**Proof**
We’ll note with \(M, N\) the intersection points with \(AB\) respectively \(AC\) of the \(A\)-mixt-linear circle adjointly ex-inscribed. We have \(\angle BCA \equiv \angle BAA'\) and \(\angle A' AB \equiv \angle A'' AM\) (see Fig.1).

Because \(\angle A'' AM = \angle ANM\), we obtain \(\angle ANM \equiv \angle ACB\) which implies that \(MN\) is parallel to \(BC\).

**Proposition 3**
The radius $R_A$ of the $A$-mixt-linear circle adjointly ex-inscribed to triangle $ABC$ is given by the following formula

$$R_A = \frac{4(p-b)(p-c)R}{(b-c)^2}$$

Proof
The sinus theorem in the triangle $AMN$ implies

$$R_A = \frac{MN}{2\sin A}$$

We observe that the triangles $AMN$ and $ABC$ are similar; it results that

$$\frac{MN}{a} = \frac{AM}{c}.$$  

Considering the power of the point $B$ in rapport to the $A$-mixt-linear circle adjointly ex-inscribed of triangle $ABC$, we obtain

$$BA \cdot BM = BD^2.$$

From the theorem of the external bisectrix we have

$$D'B = \frac{ac}{b-c}.$$  

We obtain then

$$BM = \frac{a^2c}{(b-c)^2},$$

therefore

$$AM = \frac{c(a-b+c)(a+b-c)}{(b-c)^2} = \frac{4c(p-b)(p-c)}{(b-c)^2}.$$  

and

$$MN = \frac{4a(p-b)(p-c)}{(b-c)^2}.$$  

From the sinus theorem applied in the triangle $ABC$ results that

$$\frac{a}{2\sin A} = R$$

and we obtain that

$$R_A = \frac{4(p-b)(p-c)R}{(b-c)^2}.$$

Remark
If we note $P \in L_A \cap AD'$ and $AD' = l_a'$ (the length of the exterior bisectrix constructed from $A$) in triangle $L_A PA'$, we find

$$R_A = \frac{l_a'}{2\sin \frac{B-C}{2}}.$$  

We’ll remind here several results needed for the remaining of this presentation.

Definition 2
We define an adjointly circle of triangle $ABC$ a circle which contains two vertexes of the triangle and in one of these vertexes is tangent to the respective side.
**Theorem 1**

The adjointly circles $\overline{AB}, \overline{BC}, \overline{CA}$ have a common point $\Omega$; similarly, the circles $\overline{BA}, \overline{CB}, \overline{AC}$ have a common point $\Omega'$.

The points $\Omega$ and $\Omega'$ are called the points of Brocard: $\Omega$ is the direct point of Brocard and $\Omega'$ is called the retrograde point.

The points $\Omega$ and $\Omega'$ are conjugate isogonal

\[
\angle \Omega AB = \angle \Omega BC = \angle \Omega CA = \omega \\
\angle \Omega AC = \angle \Omega' CB = \angle \Omega' BA = \omega
\]

(see Fig. 2).

The angle $\omega$ is called the Brocard angle. More information can be found in [3].

![Figure 2](image)

**Proposition 4**

In triangle $ABC$ in which $D'$ is the leg of the external bisectrix of the angle $BAC$, the $A$-mixt-linear circle adjointly ex-inscribed to triangle $ABC$ is an adjointly circle of triangles $AD'B, AD'C$.

**Proposition 5**

In a triangle $ABC$ in which $D'$ is the leg of the external bisectrix of the angle $BAC$, the direct points of Brocard corresponding to triangles $AD'B, AD'C$, A, D' are concyclic.
The following theorems show remarkable properties of the mixt-linear circles adjointly ex-inscribed associated to a triangle $ABC$.

**Theorem 2**
The triangle $L_aL_bL_c$ determined by the centers of the mixt-linear circles adjointly ex-inscribed to triangle $ABC$ and the tangential triangle $T_aT_bT_c$ corresponding to $ABC$ are orthological. Their orthological centers are $O$ the center of the circumscribed circle to triangle $ABC$ and the radical center of the mixt-linear circles adjointly ex-inscribed associated to triangle $ABC$.

**Proof**
The perpendiculars constructed from $ABCL$, $L$, $L$ on the corresponding sides of the tangential triangle contain the radiuses $OA$, $OB$, $OC$ respectively of the circumscribed circle. Consequently, $O$ is the orthological center of triangles $L_aL_bL_c$ and $T_aT_bT_c$.

In accordance to the theorem of orthological triangles and the perpendiculars constructed from $T_a,T_b,T_c$ respectively on the sides of the triangle $L_aL_bL_c$ are concurrent. The point $T_a$ belongs to the radical axis of the circumscribed circles to triangle $ABC$ and the $C$-mixt-linear circle adjointly ex-inscribed to triangle $ABC$ (belongs to the common tangent constructed in $C$ to these circles).

On the other side $T_a$ belongs to the radical axis of the $B$ and $C$-mixt-linear circle adjointly ex-inscribed, which means that the perpendicular constructed from $T_a$ on the $L_bL_c$ centers line passes through the radical center of the mixt-linear circle adjointly ex-inscribed associated to the triangle; which is the second orthological center of the considered triangles.

**Proposition 6**
The triangle $L_aL_bL_c$ (determined by the centers of the mixt-linear circles adjointly inscribed associated to the triangle $ABC$) and the triangle $L_aL_bL_c$ (determined by the centers of the mixt-linear circles adjointly ex-inscribed associated to the triangle $ABC$) are homological. The homological center is the point $O$, which is the center of the circumscribed circle of triangle $ABC$.

The proof results from the fact that the points $L_a,A,L_a,O$ are collinear. Also, $L_b,B,L_b,O$ and $L_c,C,L_c,O$ are collinear.

**Definition 3**
Given three circles of different centers, we define their Apollonius circle as each of the circles simultaneous tangent to three given circles.

**Observation**
The circumscribed circle to the triangle $ABC$ is the Apollonius circle for the mixt-linear circles adjointly ex-inscribed associated to $ABC$. 

5
Theorem 3
The Apollonius circle which has in its interior the mixt-linear circles adjointly ex-inscribed to triangle $ABC$ is tangent with them in the points $T_1, T_2, T_3$ respectively. The lines $AT_1, BT_2, CT_3$ are concurrent.

Proof
We’ll use the D’Alembert theorem: Three circles non-congruent whose centers are not collinear have their six homothetic centers placed on four lines, three on each line.

The vertex $A$ is the homothety inverse center of the circumscribed circle $(O)$ and of the $A$-mixt-linear circle adjointly ex-inscribed $(L_A)$; $T_1$ is the direct homothety center of the Apollonius circle which is tangent to the mixt-linear circles adjointly ex-inscribed and of circle $(L_A)$, and $J$ is the center of the direct homothety of the Apollonius circle and of the circumscribed circle $(O)$.

According to D’Alembert theorem, it results that the points $A, J, T_1$ are collinear. Similarly is shown that the points $B, J, T_2$ and $C, J, T_3$ are collinear.

Consequently, $J$ is the concurrency point of the lines $AT_1, BT_2, CT_3$.