APPLICATIONS OF WALLIS THEOREM

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Abstract: In this paper we present theorems and applications of Wallis theorem related to trigonometric integrals.

Let's recall Wallis Theorem:

Theorem 1. (Wallis, 1616-1703)

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x dx = \int_0^{\frac{\pi}{2}} \cos^{2n+1} x dx = \frac{2 \cdot 4 \cdot \ldots \cdot (2n)}{1 \cdot 3 \cdot \ldots \cdot (2n+1)}.$$

Proof: Using the integration by parts, we obtain

$$I_n = \int_0^{\frac{\pi}{2}} \sin^{2n+1} x dx = \int_0^{\frac{\pi}{2}} \sin^{2n} x \sin x dx = -\cos x \cdot \sin 2nx \Big|_0^{\frac{\pi}{2}} +$$

$$+2n \int_0^{\frac{\pi}{2}} \sin^{2n+1} x \left(1 - \sin^2 x\right) dx = 2nI_{n-1} - 2nI_n$$

from where:

$$I_n = \frac{2n}{2n+1} I_{n-1}.$$

By multiplication, we obtain the statement.
We prove in the same manner for $\cos x$.

Theorem 2.

$$\int_0^{\frac{\pi}{2}} \sin^{2n} x dx = \int_0^{\frac{\pi}{2}} \cos^{2n} x dx = \frac{1 \cdot 3 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot \ldots \cdot (2n)} \cdot \frac{\pi}{2}.$$

Proof: Same as the first theorem.

Theorem 3. If $f(x) = \sum_{k=0}^{\infty} a_{2k} x^{2k}$, then

$$\int_0^{\frac{\pi}{2}} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f(\cos x) dx = \frac{\pi}{2} a_0 + \frac{\pi}{2} \sum_{k=1}^{\infty} a_{2k} \frac{1 \cdot 3 \cdot \ldots \cdot (2k-1)}{2 \cdot 4 \cdot \ldots \cdot (2k)}.$$

Proof: In the function $f(x)=\sum_{k=0}^{\infty} a_{2k}x^{2k}$ we substitute x by $\sin x$ and then integrate from 0 to $\frac{\pi}{2}$, and we use the second theorem.

Theorem 4. If $g(x)=\sum_{k=0}^{\infty} a_{2k+1}x^{2k+1}$, then

$$\int_0^{\frac{\pi}{2}} g(\sin x)dx=\int_0^{\frac{\pi}{2}} g(\cos x)dx=a_1+\sum_{k=1}^{\infty} a_{2k+1}\frac{2\cdot 4\cdot\ldots\cdot(2k)}{1\cdot 3\cdot\ldots\cdot(2k+1)}.$$

Theorem 5. If $h(x)=\sum_{k=0}^{\infty} a_k x^k$, then

$$\int_0^{\frac{\pi}{2}} h(\sin x)dx=\int_0^{\frac{\pi}{2}} h(\cos x)dx=\frac{\pi}{2}a_0+a_1+\sum_{k=1}^{\infty}\left(\frac{\pi}{2}a_{2k}\frac{1\cdot 3\cdot\ldots\cdot(2k-1)}{2\cdot 4\cdot\ldots\cdot(2k)}+\right.$$
$$\left.+a_{2k+1}\frac{2\cdot 4\cdot\ldots\cdot(2k)}{1\cdot 3\cdot\ldots\cdot(2k+1)}\right).$$

Application 1.

$$\int_0^{\frac{\pi}{2}} \sin(\sin x)dx=\int_0^{\frac{\pi}{2}} \sin(\cos x)dx=\sum_{k=0}^{\infty}(-1)^k\frac{1}{1^2\cdot 3^2\cdot\ldots\cdot(2k+1)^2}$$

Proof: We use that $\sin x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$.

Application 2.

$$\int_0^{\frac{\pi}{2}} \cos(\sin x)dx=\int_0^{\frac{\pi}{2}} \cos(\cos x)dx=\frac{\pi}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{4^k(k!)^2}.$$

Proof: We use that $\cos x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}$.

Application 3.

$$\int_0^{\frac{\pi}{2}} sh(\sin x)dx=\int_0^{\frac{\pi}{2}} sh(\cos x)dx=\sum_{k=0}^{\infty}\frac{1}{1^2\cdot 3^2\cdot\ldots\cdot(2k+1)^2}.$$

Proof: We use that $shx=\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}$

Application 4.

$$\int_0^{\frac{\pi}{2}} ch(\sin x)dx = \int_0^{\frac{\pi}{2}} ch(\cos x)dx = \frac{\pi}{2}\sum_{k=0}^{\infty}\frac{1}{4^k (k!)^2}.$$

Proof: We use that $chx = \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}$.

Application 5.

$$\sum_{k=1}^{\infty}\frac{1}{k^2}\frac{\pi^2}{6}$$

Proof: In the expression of $\arcsin x = x + \sum_{k=1}^{\infty}\frac{1\cdot 3\cdot\ldots\cdot(2k-1)x^{2k+1}}{2\cdot 4\cdot\ldots\cdot(2k)(2k+1)}$ we substitute x by $\sin x$, and use theorem 4. It results that $\frac{\pi^2}{8} = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$.

Because:

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} + \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}$$

we obtain:

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi}{6}.$$

Application 6.

$$\int_0^{\frac{\pi}{2}} \sin x \ \mathrm{ctg}(\sin x)dx = \int_0^{\frac{\pi}{2}} \cos x \ \mathrm{ctg}(\cos x)dx = \frac{\pi}{2} - \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{B_k}{(k!)^2}$$

where B_k is the k-th Bernoulli type number (see [1]).

Proof: We use that $x\mathrm{ctg}x = 1 - \sum_{k=1}^{\infty}\frac{4^k B_k}{(2k)!}x^{2k}$.

Application 7.

$$\int_0^{\frac{\pi}{2}} arctg(\sin x)dx = \int_0^{\frac{\pi}{2}} arctg(\cos x)dx = 1 + \sum_{k=1}^{\infty}(-1)^k \frac{2\cdot 4\cdot\ldots\cdot(2k)}{1\cdot 3\cdot\ldots\cdot(2k-1)(2k+1)^2}.$$

Proof: We use that $arctgx = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}$.

Application 8.

$$\int_0^{\frac{\pi}{2}} \arg th(\sin x)dx = \int_0^{\frac{\pi}{2}} \arg th(\cos x)dx = 1 + \sum_{k=1}^{\infty}\frac{2\cdot 4\cdot\ldots\cdot(2k)}{1\cdot 3\cdot\ldots\cdot(2k-1)(2k+1)^2}.$$

Proof: We use that $\arg th\ x = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}$.

Application 9.

$$\int_0^{\frac{\pi}{2}} \arg sh(\sin x)dx = \int_0^{\frac{\pi}{2}} \arg sh(\cos x)dx = \sum_{k=1}^{\infty}(-1)^k \frac{1}{(2k+1)^2}.$$

Proof: We use that $\arg shx = \sum_{k=0}^{\infty}(-1)^k \frac{1\cdot 3\cdot \ldots \cdot (2k-1)x^{2k+1}}{2\cdot 4\cdot \ldots \cdot (2k)(2k+1)}$.

Application 10.

$$\int_0^{\frac{\pi}{2}} \mathrm{tg}(\sin x)dx = \int_0^{\frac{\pi}{2}} \mathrm{tg}(\cos x)dx = \sum_{k=1}^{\infty} \frac{2^{2k-1}(4^k-1)B_k}{1^2\cdot 3^2\cdot \ldots \cdot (2k-1)^2 k}.$$

Proof: We use that $\mathrm{tg}\, x = \sum_{k=1}^{\infty} \frac{2^{2k}(4^k-1)B_k}{(2k)!} x^{2k-1}$.

Application 11.

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin(\sin x)}dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin(\cos x)}dx = \frac{\pi}{2} + \pi\sum_{k=1}^{\infty} \frac{\left(2^{2k-1}-1\right)B_k}{2^{2k}(k!)^2}$$

Proof: We use that $\frac{x}{\sin x} = 1 + 2\sum_{k=1}^{\infty} \frac{\left(2^{2k-1}-1\right)B_k}{(2k)!} x^{2k}$.

Application 12.

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{sh(\sin x)}dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{sh(\cos x)}dx = \frac{\pi}{2} + \pi\sum_{k=1}^{\infty} \frac{\left(2^{2k-1}-1\right)B_k}{2^{2k}(k!)^2}.$$

Proof: We use that $\frac{x}{shx} = 1 + 2\sum_{k=1}^{\infty}(-1)^k \frac{\left(2^{2k-1}-1\right)B_k}{(2k)!} x^{2k}$.

Application 13.

$$\int_0^{\frac{\pi}{2}} \sec(\sin x)dx = \int_0^{\frac{\pi}{2}} \sec(\cos x)dx = \frac{\pi}{2} + \pi\sum_{k=1}^{\infty} \frac{E_k}{2^{2k+1}(k!)^2},$$

where E_k is the k-th Euler type number (see [1]).

Proof: We use that $\sec x = 1 + \sum_{k=1}^{\infty} \frac{E_k}{(2k)!} x^{2k}$

Application 14.

$$\int_0^{\frac{\pi}{2}} \sec h(\sin x)dx = \int_0^{\frac{\pi}{2}} \sec h(\cos x)dx = \frac{\pi}{2} + \pi\sum_{k=1}^{\infty}(-1)^k \frac{E_k}{2^{2k+1}(k!)^2}.$$

Proof: We use that $\sec h\ x = 1 + \sum_{k=1}^{\infty}(-1)^k \frac{E_k}{(2k)!} x^{2k}$.

REFERENCES

[1] Octav Mayer – Theoria funcţiilor de o variabilă complexă – Ed. Academiei, Bucharest, 1981.

[2] Mihály Bencze – About Taylor formula – (manuscript).

[Published in "Octogon", Vol. 6, No. 2, 117-120, 1998.]