An Application of a Theorem of Orthohomological Triangles

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Abstract.
In this note we prove a problem given at a Romanian student mathematical competition, and we obtain an interesting result by using a *Theorem of Orthohomological Triangles*[1].

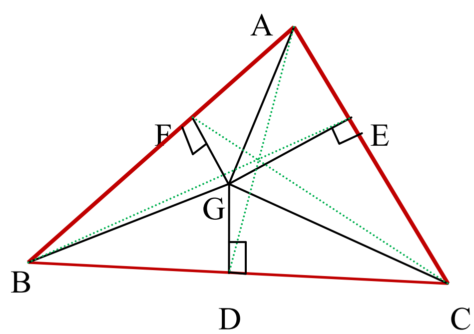
Problem L. 176 (from [1])

Let D, E, F be the projections of the centroid G of the triangle ABC on the lines BC, CA, and respectively AB. Prove that the Cevian lines AD, BE, and CF meet in an unique point if and only if the triangle is isosceles. {Proposed by Temistocle Bîrsan.}

Proof

Applying the generalized Pythagorean theorem in the triangle BGC, we obtain:

$$CG^2 = BG^2 + BC^2 - 2BD \cdot BC \quad (1)$$



Because $CG = \frac{2}{3}m_c$, $BG = \frac{2}{3}m_b$ and from the median's theorem it results:

$$4m_b^2 = 2(a^2 + c^2) - b^2 \text{ and } 4m_c^2 = 2(a^2 + b^2) - c^2$$

From (1) we get: $BD = \dfrac{3a^2 - b^2 + c^2}{6a}$.

[1] It has been called the *Smarandache-Pătraşcu Theorem of Orthohomological Triangles* (see [2], [3], [4]).

From $BC = a$ and $BC = BD + DC$, we get that:

$$DC = \frac{3a^2 + b^2 - c^2}{6a}$$

Similarly we find:

$$CE = \frac{3b^2 - c^2 + a^2}{6b},\ EA = \frac{3b^2 + c^2 - a^2}{6b}$$

$$FA = \frac{3c^2 - a^2 + b^2}{6c},\ FB = \frac{3c^2 + a^2 - b^2}{6c}.$$

Applying Ceva's theorem it results that AD, BE, CF are concurrent if and only if

$$\left(3a^2 - b^2 + c^2\right)\left(3b^2 - c^2 + a^2\right)\left(3c^2 - a^2 + b^2\right) = \left(3a^2 + b^2 - c^2\right)\left(3b^2 + c^2 - a^2\right)\left(3c^2 + a^2 - b^2\right) \quad (2)$$

Let's consider the following notations:

$$a^2 + b^2 + c^2 = T,\ 2a^2 - 2b^2 = \alpha,\ 2b^2 - 2c^2 = \beta,\ 2c^2 - 2a^2 = \gamma$$

From (2) it results:

$$\left(T + \alpha\right)\left(T + \beta\right)\left(T + \gamma\right) = \left(T - \alpha\right)\left(T - \beta\right)\left(T - \gamma\right).$$

And from here:

$$T^3 + \left(\alpha + \beta + \gamma\right)T^2 + \left(\alpha\beta + \beta\gamma + \gamma\alpha\right)T + \alpha\beta\gamma = T^3 - \left(\alpha + \beta + \gamma\right)T^2 + \left(\alpha\beta + \beta\gamma + \gamma\alpha\right)T - \alpha\beta\gamma.$$

Because $\alpha + \beta + \gamma = 0$, we obtain that $2\alpha\beta\gamma = 0$, therefore $\alpha = 0$ or $\beta = 0$ or $\gamma = 0$, thus $a = b$ or $b = c$ or $a = c$; consequently the triangle ABC is isosceles.

The reverse: If ABC is an isosceles triangle, then it is obvious that AD, BE, and CF are concurrent.

Observations

1. The proved problem asserts that:
   "A triangle ABC and the pedal triangle of its weight center are orthomological triangles if and only if the triangle ABC is isosceles."

2. Using the previous result and the Smarandache-Pătrăşcu Theorem (see [2], [3], [4]) we deduce that:
   "A triangle ABC and the pedal triangle of its simedian center are orthomological triangles if and only if the triangle ABC is isosceles."

References

[1] Temistocle Bîrsan, Training problems for mathematical contest, B. College Level – L. 176, Recreaţii Matematice journal, Iași, Romania, Year XII, No. 1, 2010.

[2] Ion Pătrașcu & Florentin Smarandache, *A Theorem about Simultaneous Orthological and Homological Triangles*, in arXiv.org, Cornell University, NY, USA.

[3] Mihai Dicu, *The Smarandache- Pătrașcu Theorem of Orthohomological Triangles*, http://www.scribd.com/doc/28311880/Smarandache-Patrascu-Theorem-of-Orthohomological-Triangles.
[4] Claudiu Coandă, *A Proof in Barycentric Coordinates of the Smarandache-Pătrașcu Theorem*, Sfera journal, 2010.