A Class of Orthohomological Triangles

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Abstract.
In this article we propose to determine the triangles’ class $A_iB_iC_i$ orthohomological with a given triangle ABC, inscribed în the triangle ABC ($A_i \in BC,\ B_i \in AC,\ C_i \in AB$).

We’ll remind, here, the fact that if the triangle $A_iB_iC_i$ inscribed in ABC is orthohomologic with it, then the perpendiculars in A_i, B_i, respectively in C_i on BC, CA, respectively AB are concurrent in a point P_i (the orthological center of the given triangles), and the lines AA_i, BB_i, CC_i are concurrent in point (the homological center of the given triangles).

To find the triangles $A_iB_iC_i$, it will be sufficient to solve the following problem.

Problem.
Let’s consider a point P_i in the plane of the triangle ABC and $A_iB_iC_i$ its pedal triangle. Determine the locus of point P_i such that the triangles ABC and $A_iB_iC_i$ to be homological.

Solution.
Let’s consider the triangle ABC, $A(1,0,0),\ B(0,1,0),\ C(0,0,1)$, and the point $P_i(\alpha,\beta,\gamma),\ \alpha+\beta+\gamma=0$.

The perpendicular vectors on the sides are:

$$U_{BC}^{\perp}\left(2a^2,\ -a^2-b^2+c^2,\ -a^2+b^2-c^2\right)$$
$$U_{CA}^{\perp}\left(-a^2-b^2+c^2,\ 2b^2,\ a^2-b^2-c^2\right)$$
$$U_{AB}^{\perp}\left(-a^2+b^2-c^2,\ a^2-b^2-c^2,\ 2c^2\right)$$

The coordinates of the vector $\overrightarrow{BC}$ are $(0,-1,1)$, and the line BC has the equation $x=0$.

The equation of the perpendicular raised from point P_i on BC is:

$$\begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ 2a^2 & -a^2-b^2+c^2 & -a^2+b^2-c^2 \end{vmatrix}=0$$

We note $A_i(x,y,z)$, because $A_i \in BC$ we have:

$x=0$ and $y+z=1$.

The coordinates y and z of A_i can be found by solving the system of equations

$$\begin{cases} \begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ 2a^2 & -a^2-b^2+c^2 & -a^2+b^2-c^2 \end{vmatrix} = 0 \\ y+z=0 \end{cases}$$

We have:

$$y \cdot \begin{vmatrix} \alpha & \gamma \\ 2a^2 & -a^2+b^2-c^2 \end{vmatrix} = z \cdot \begin{vmatrix} \alpha & \beta \\ 2a^2 & -a^2-b^2+c^2 \end{vmatrix},$$

$$y\left[\alpha\ \left(-a^2+b^2-c^2\right)-2\gamma a^2\right] = z\left[\alpha\ \left(-a^2-b^2+c^2\right)-2\beta a^2\right],$$

$$y + y \cdot \frac{\alpha\ \left(-a^2+b^2-c^2\right)-2\gamma a^2}{\alpha\ \left(-a^2-b^2+c^2\right)-2\beta a^2} = 1,$$

$$y \cdot \frac{\alpha\ \left(-a^2-b^2+c^2\right)-2\beta a^2+\alpha\ \left(-a^2+b^2-c^2\right)-2\gamma a^2}{\alpha\ \left(-a^2-b^2+c^2\right)-2\beta a^2} = 1,$$

$$y \cdot \frac{-2a^2\left(\alpha+\beta+\gamma\right)}{\alpha\left(-a^2-b^2+c^2\right)-2\beta a^2} = 1,$$

it results

$$y = \frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right)+\beta$$

$$z = 1-y = 1-\beta-\frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right) = \alpha+\gamma-\frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right).$$

Therefore,

$$A_i\left(0,\ \ \frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right)+\beta,\ \ \frac{\alpha}{2a^2}\left(a^2-b^2+c^2\right)+\gamma\right).$$

Similarly we find:

$$B_i\left(\frac{\beta}{2b^2}\left(a^2+b^2-c^2\right)+\alpha,\ 0,\ \ \frac{\beta}{2b^2}\left(-a^2+b^2+c^2\right)+\gamma\right),$$

$$C_i\left(\frac{\gamma}{2c^2}\left(a^2-b^2+c^2\right)+\alpha,\ \frac{\gamma}{2c^2}\left(-a^2+b^2+c^2\right)+\beta,\ \ 0\right).$$

We have:

$$\frac{\overrightarrow{A_iB}}{\overrightarrow{A_iC}} = -\frac{\frac{\alpha}{2a^2}\left(a^2-b^2+c^2\right)+\gamma}{\frac{\alpha}{2a^2}\left(a^2+b^2-c^2\right)+\beta} = -\frac{\alpha c\cos B+\gamma a}{\alpha b\cos C+\beta a}$$

$$\frac{\overrightarrow{B_iC}}{\overrightarrow{B_iA}} = -\frac{\frac{\beta}{2b^2}\left(a^2+b^2-c^2\right)+\alpha}{\frac{\alpha}{2a^2}\left(-a^2+b^2+c^2\right)+\gamma} = -\frac{\beta a\cos C+\alpha b}{\beta c\cos A+\gamma b}.$$

$$\frac{\overrightarrow{C_iA}}{\overrightarrow{C_iB}} = -\frac{\frac{\gamma}{2c^2}\left(-a^2+b^2+c^2\right)+\beta}{\frac{\gamma}{2c^2}\left(a^2-b^2+c^2\right)+\alpha} = -\frac{\gamma b\cos A+\beta c}{\gamma a\cos B+\alpha c}$$

(We took into consideration the cosine's theorem: $a^2 = b^2+c^2-2bc\cos A$).
In conformity with Ceva's theorem, we have:

$$\frac{\overrightarrow{A_iB}}{\overrightarrow{A_iC}}\cdot\frac{\overrightarrow{B_iC}}{\overrightarrow{B_iA}}\cdot\frac{\overrightarrow{C_iA}}{\overrightarrow{C_iB}} = -1.$$

$$(a\gamma+\alpha c\cos B)(b\alpha+\beta a\cos C)(c\beta+\gamma b\cos A) =$$
$$=(a\beta+\alpha b\cos C)(b\gamma+\beta c\cos A)(c\alpha+\gamma a\cos B)$$
$$a\alpha\left(b^2\gamma^2-c^2\beta^2\right)(\cos A-\cos B\cos C)+b\beta\left(c^2\alpha^2-a^2\gamma^2\right)(\cos B-\cos A\cos C)+$$
$$+c\gamma\left(a^2\beta^2-b^2\alpha^2\right)(\cos C-\cos A\cos B)=0.$$

Dividing it by $a^2b^2c^2$, we obtain that the equation in barycentric coordinates of the locus $\mathfrak{L}$ of the point P_i is:

$$\frac{\alpha}{a}\left(\frac{\gamma^2}{c^2}-\frac{\beta^2}{b^2}\right)(\cos A-\cos B\cos C)+\frac{\beta}{b}\left(\frac{\alpha^2}{a^2}-\frac{\gamma^2}{c^2}\right)(\cos B-\cos A\cos C)+$$
$$+\frac{\gamma}{c}\left(\frac{\beta^2}{b^2}-\frac{\alpha^2}{a^2}\right)(\cos C-\cos A\cos B)=0.$$

We note $\overline{d}_A$, $\overline{d}_B$, $\overline{d}_C$ the distances oriented from the point P_i to the sides BC, CA respectively AB, and we have:

$$\frac{\alpha}{a}=\frac{\overline{d}_A}{2s},\ \frac{\beta}{b}=\frac{\overline{d}_B}{2s},\ \frac{\gamma}{c}=\frac{\overline{d}_C}{2s}.$$

The locus' $\mathfrak{L}$ equation can be written as follows:

$$\overline{d}_A\left(\overline{d}_C^2-\overline{d}_B^2\right)(\cos A-\cos B\cos C)+\overline{d}_B\left(\overline{d}_A^2-\overline{d}_C^2\right)(\cos B-\cos A\cos C)+$$
$$+\overline{d}_C\left(\overline{d}_B^2-\overline{d}_A^2\right)(\cos C-\cos A\cos B)=0$$

Remarks.

1. It is obvious that the triangle's ABC orthocenter belongs to locus $\mathcal{L}$. The orthic triangle and the triangle ABC are orthohomologic; a orthological center is the orthocenter H, which is the center of homology.
2. The center of the inscribed circle in the triangle ABC belongs to the locus $\mathcal{L}$, because $\overline{d}_A = \overline{d}_B = \overline{d}_C = r$ and thus locus' equation is quickly verified.

Theorem (Smarandache-Pătraşcu).

If a point P belongs to locus $\mathcal{L}$, then also its isogonal P' belongs to locus $\mathcal{L}$.

Proof.

Let $P(\alpha,\beta,\gamma)$ a point that verifies the locus' $\mathcal{L}$ equation, and $P'(\alpha',\beta',\gamma')$ its isogonal in the triangle ABC. It is known that $\frac{\alpha\alpha'}{a^2} = \frac{\beta\beta'}{b^2} = \frac{\gamma\gamma'}{c^2}$. We'll prove that $P' \in \mathcal{L}$, i.e.

$$\sum \frac{\alpha'}{a}\left(\frac{\gamma'^2}{c^2} - \frac{\beta'^2}{b^2}\right)(\cos A - \cos B \cos C) = 0$$

$$\sum \frac{\alpha'}{a}\left(\frac{\gamma'^2 b^2 - \beta'^2 c^2}{b^2 c^2}\right)(\cos A - \cos B \cos C) = 0$$

$$\sum \frac{\alpha'}{ab^2c^2}\left(\gamma'^2 b^2 - \beta'^2 c^2\right)(\cos A - \cos B \cos C) = 0 \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{\alpha'}{ab^2c^2}\left(\frac{\gamma'\beta\beta' c^2}{\gamma} - \frac{c^2\gamma\gamma'\beta'}{\beta}\right)(\cos A - \cos B \cos C) = 0 \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{\alpha'\beta'\gamma'}{ab^2c^2}\left(\frac{\beta c^2}{\gamma} - \frac{\gamma b^2}{\beta}\right)(\cos A - \cos B \cos C) = 0 \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{\alpha'\beta'\gamma'}{ab^2c^2}\left(\frac{\beta^2 c^2 - \gamma^2 b^2}{\beta\gamma}\right)(\cos A - \cos B \cos C) = 0 \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{\alpha}{a}\left(\frac{\alpha'\beta'\gamma'}{\alpha\beta\gamma}\right)\frac{1}{b^2c^2}\cdot b^2c^2\left(\frac{\beta^2}{b^2} - \frac{\gamma^2}{c^2}\right)(\cos A - \cos B \cos C) = 0.$$

We obtain that:

$$\frac{\alpha'\beta'\gamma'}{\alpha\beta\gamma}\sum \frac{\alpha}{a}\left(\frac{\gamma^2}{c^2} - \frac{\beta^2}{b^2}\right)(\cos A - \cos B \cos C) = 0,$$

this is true because $P \in \mathcal{L}$.

Remark.

We saw that the triangle 's ABC orthocenter H belongs to the locus, from the precedent theorem it results that also O, the center of the circumscribed circle to the triangle ABC (isogonable to H), belongs to the locus.

Open problem:

What does it represent from the geometry's point of view the equation of locus $\mathcal{L}$?

In the particular case of an equilateral triangle we can formulate the following:

Proposition:

The locus of the point P from the plane of the equilateral triangle ABC with the property that the pedal triangle of P and the triangle ABC are homological, is the union of the triangle's heights.

Proof:

Let $P(\alpha, \beta, \gamma)$ a point that belongs to locus $\mathfrak{L}$. The equation of the locus becomes:

$$\alpha\left(\gamma^2-\beta^2\right)+\beta\left(\alpha^2-\gamma^2\right)+\gamma\left(\beta^2-\alpha^2\right)=0$$

Because:

$$\alpha\left(\gamma^2-\beta^2\right)+\beta\left(\alpha^2-\gamma^2\right)+\gamma\left(\beta^2-\alpha^2\right)=\alpha\gamma^2-\alpha\beta^2+\beta\alpha^2-\beta\gamma^2+\gamma\beta^2-\gamma\alpha^2=$$
$$=\alpha\beta\gamma+\alpha\gamma^2-\alpha\beta^2+\beta\alpha^2-\beta\gamma^2+\gamma\beta^2-\gamma\alpha^2-\alpha\beta\gamma=$$
$$=\alpha\beta(\gamma-\beta)+\alpha\gamma(\gamma-\beta)-\alpha^2(\gamma-\beta)-\beta\gamma(\gamma-\beta)=$$
$$=(\gamma-\beta)\left[\alpha(\beta-\alpha)-\gamma(\beta-\alpha)\right]=(\beta-\alpha)(\alpha-\gamma)(\gamma-\beta).$$

We obtain that $\alpha=\beta$ or $\beta=\gamma$ or $\gamma=\alpha$, that shows that P belongs to the medians (heights) of the triangle ABC.

References:

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