THE DUAL OF THE ORTHOPOLE THEOREM

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Abstract

In this article we prove the theorems of the orthopole and we obtain, through duality, its dual, and then some interesting specific examples of the dual of the theorem of the orthopole.

The transformation through duality was introduced in 1822 by the French mathematician Victor Poncelet. By the duality in rapport with a given circle to the points correspond lines (their polars), and to the straight lines correspond points (their poles).

Given a figure *F* formed of lines, points and, eventually, a circle, by applying to it the transformation through duality in rapport with the circle, we obtain a new figure F', which is formed of lines that are the polars of the figure's *F* points in rapport with the circle and from points that are the poles of the figure's *F* lines in rapport with the circle. Also, through duality to a given theorem corresponds a new theorem called its dual. After this introduction, we'll obtain the dual of the orthopole theorem.

The Orthopole Theorem (Soons – 1886).

If ABC is a triangle, d a line in its plane and A', B', C' the vertexes' projections of A, B, C on d, then the perpendiculars from A', B', C' on the sides BC, CA, AB are concurrent (the concurrence point is called the triangle's orthopole, in rapport to the line d).

In order to proof the orthopole's theorem will be using the following:

Theorem (L. Carnot - 1803)

The necessary and sufficient condition that the perpendiculars drawn on the sides BC, CA, AB of the triangle ΔABC, through the points A_1, B_1, C_1 that belong to these sides, to be concurrent is:

$$A_1B^2 - A_1C^2 + B_1C^2 - B_1A^2 + C_1A^2 - C_1B^2 = 0.$$

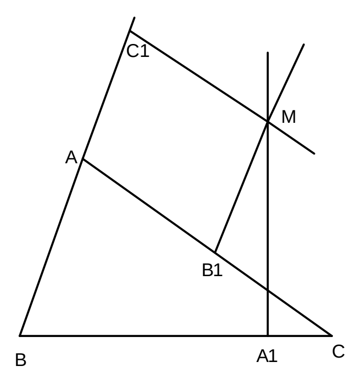

Fig. 1

Proof:

The condition is **necessary**: Let M be the concurrent point of the perpendiculars drawn in

A_1, B_1, C_1 on the sides of the triangle ΔABC (see Fig. 1).

We have

$$A_1B^2 - A_1C^2 = MB^2 - MA_1^2 - MC^2 + MA_1^2 = MB^2 - MC^2$$
$$B_1C^2 - B_1A^2 = MC^2 - MB_1^2 - MB_1^2 - MA^2 = MC^2 - MA^2$$
$$C_1A^2 - C_1B^2 = MA^2 - MC_1^2 + MC_1^2 - MB^2 = MA^2 - MB^2$$

Adding member by member these three relations it is obtained the relation from the above theorem.

The condition is **<u>sufficient</u>**: Let M be the intersection of the perpendiculars in A_1 on BC and in B_1 on AC, şi $C_1^{'}$ the projection of M on AB.

We have:

$$A_1B^2 - A_1C^2 + B_1C^2 - B_1A^2 + C_1^{'}A^2 - C_1^{'}B^2 = 0,$$

and from hypothesis:

$$A_1B^2 - A_1C^2 + B_1C^2 - B_1A^2 + C_1A^2 - C_1B^2 = 0.$$

We obtain:

$$C_1^{'}A^2 - C_1^{'}B^2 = C_1A^2 - C_1B^2,$$

from which we find: $C_1^{'} = C_1$, and therefore, the perpendiculars drawn in A_1, B_1, C_1 on the triangle's sides are concurrent.

<u>The proof of the Orthopole Theorem</u>

Let's note A_1, B_1, C_1 the projections of the points $A_1^{'}, B_1^{'}, C_1^{'}$ on BC, CA, AB (see Fig. 2).

We have:

$$A_1B^2 - A_1C^2 = A^{'}B^2 - A^{'}C^2 = BB^{'2} + A^{'}B^{'2} - CC^{'2} - A^{'}C^{'2} \quad (1).$$

Similarly, we obtain:

$$B_1C^2 - B_1A^2 = B^{'}C^2 - B^{'}A^2 = B^{'}C^{'2} + CC^{'2} - A^{'}B^{'2} - AA^{'2} \quad (2).$$
$$C_1A^2 - C_1B^2 = C^{'}A^2 - C^{'}B^2 = AA^{'2} + A^{'}C^{'2} - B^{'}C^{'2} - BB^{'2} \quad (3).$$

From the relations (1), (2) and (3), we obtain:

$$A_1B^2 - A_1C^2 + B_1C^2 - B_1A^2 + C_1A^2 - C_1B^2 = 0,$$

relation that in conformity to the Carnot's Theorem implies the concurrency of the lines $A^{'}A_1, B^{'}B_1, C^{'}C_1$.

We denote with O the orthopole of the line d in rapport to the triangle ΔABC.
We'll apply now a duality in rapport to the circle $C(O, r)$ to the corresponding configuration of the orthopole theorem. Then, to the points A, B, C will correspond their polars a, b, c. To the line AB corresponds its pole, which we'll note $C^{'}$ and it is $a \cap b$,

similarly, we'll obtain the poles B' and A' of the lines AC and BC. To the line *d* will correspond, through the considered duality, its pole, which we'll note with *P*.

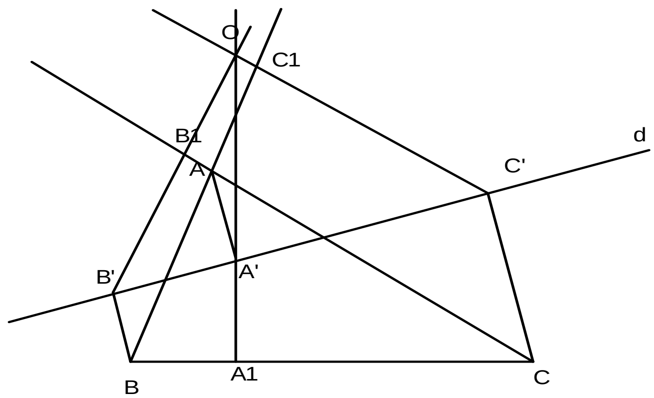

Fig. 2

If we denote with A_1', B_1', C_1' respectively, the intersections of line *P* with the sides of the triangle ΔABC, through the considered duality to these points correspond the lines $A'P, B'P$ and $C'P$ respectively. Because the lines AA' and *d* are perpendicular, their poles P_1 and P will be placed such that $m(P_1OP) = 90^\circ$, therefore P_1 is the intersection of the perpendicular in O on OP with $B'C' = a$. Similarly, the pole of the perpendicular BB' on *d* will be P_2 the intersection with $b = A'C'$ of the perpendicular drawn in O on OP and at the perpendicular's intersection in O on OP with $c = A'B'$ we will find P_3 the pole of CC'.

To the perpendicular drawn in A' on BC corresponds, through duality, its pole A_1 which is located at the intersection of the perpendicular in O on $A'O$ with PP_1. Similarly we construct the points B_1, C_1 corresponding to the perpendiculars drawn from B' on AC and from C' on AB. Because these last perpendiculars are concurrent în the line's orthopole, their poles A_1, B_1, C_1 are collinear points (they belong to the orthopole's polar).

Selecting certain points, we can formulate the following:

<u>The Dual Theorem of the Orthopole</u>

If ABC is a triangle, O and P two distinct point in its plane such that the perpendicular in O on OP intersects BC, CA, AB respectively in the points P_1, P_2, P_3, and the perpendiculars drawn in the point O on OA, OB, OC intersect respectively the lines PP_1, PP_2, PP_3 in the points A_1, B_1, C_1, then the points A_1, B_1, C_1 are collinear.

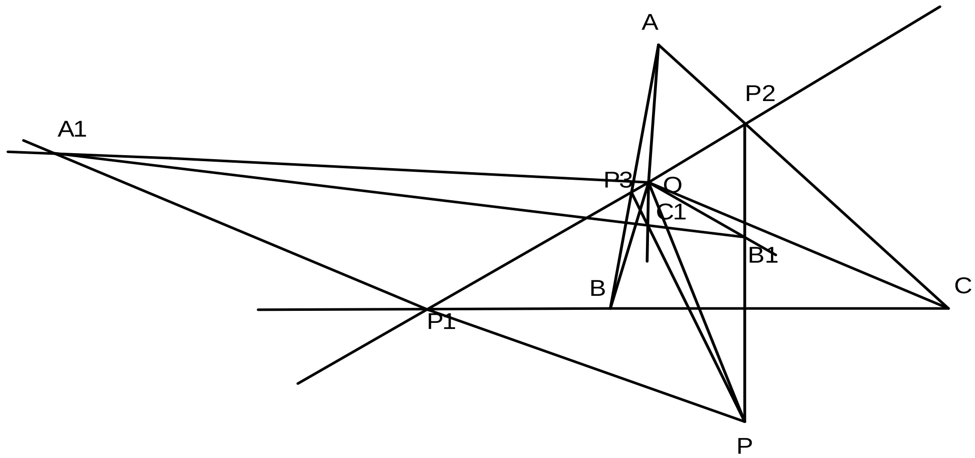

Fig. 3

Observation:

By inversing the solutions of O and P will find, following the same constructions indicated in the dual theorem of the orthopole, other collinear points A_1', B_1', C_1'.

Next, will point out several particular cases of the dual theorem of the orthopole.

1. Theorem of Bobillier

If ABC is a triangle and O is an arbitrary point in its plane, the perpendiculars drawn in O on AO, BO, CO intersect respectively BC, AC, AB into the collinear points A_1, B_1, C_1.

Proof

We apply the dual theorem of the orthopole in the particular case $P = A$: then the point P_1 coincides with A_1 because PP_1 becomes AP_1 (the point P_1 belongs to the line BC), similarly, the points B_1 and C_1 belong to AC respectively AB, it results that A_1, B_1, C_1 are collinear.

Remark

The Bobillier's Theorem was obtained transforming through duality in rapport with a circle O the theorem relative to a triangle's altitudes' concurrence.

2. Theorem

If ABC is a triangle and P a point on its circumscribed circle with the center O, the tangents in P to the circle intersect the sides BC, CA, AB respectively in P_1, P_2, P_3.

Will denote with $A^{'}, B^{'}, C^{'}$ the opposite diameters to A, B, C in the circle O and let's consider $\{A_1\} = A^{'}P \cap OP_1$, $\{B_1\} = B^{'}P \cap OP_2, \{C_1\} = C^{'}P \cap OP_3$, then the points A_1, B_1, C_1 are collinear.

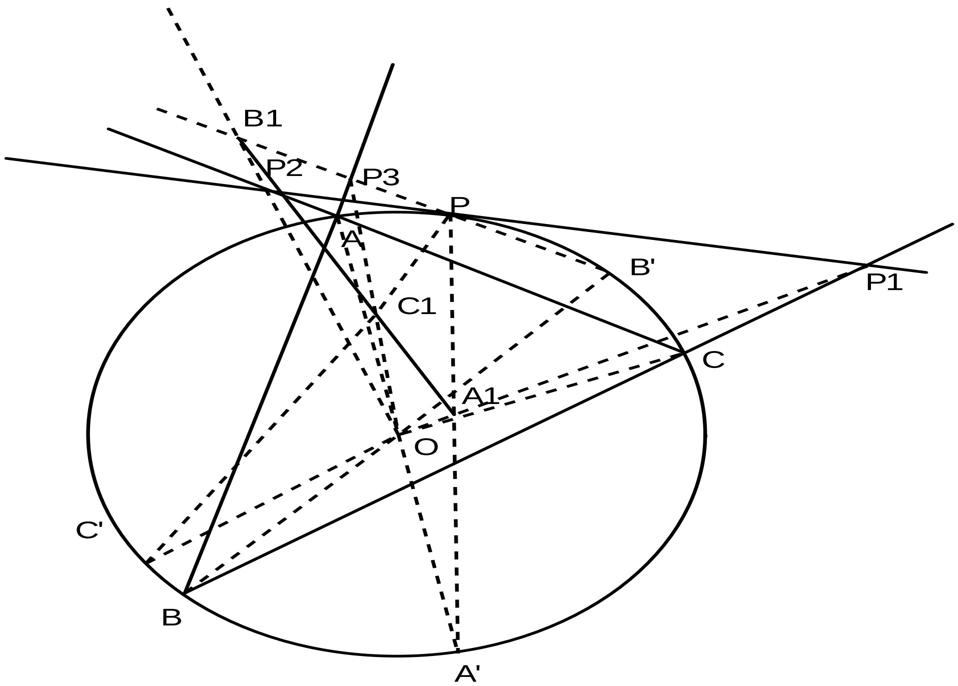

Fig. 4

Proof

The tangent in P to the circumscribed circle is perpendicular on the ray OP, therefore the points P_1, P_2, P_3 are constructed as in the hypothesis of the dual theorem of the orthopole. The point $A^{'}$ being diametric – opposite to A (see Fig. 4), we have $\mathrm{m}(APA') = 90^{\circ}$, therefore A_1 is the intersection of the perpendicular in P on AP with OP_1, similarly there are constructed B_1 and C_1, and from the dual theorem of the orthopole it results their colinearity.

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