Fusion of Masses Defined on Infinite Countable Frames of Discernment

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Abstract.
In this paper we introduce for the first time the fusion of information on infinite discrete frames of discernment and we give general results of the fusion of two such masses using the Dempster's rule and the PCR5 rule for Bayesian and non-Bayesian cases.

Introduction.

Let $\theta = \{x_1, x_2, ..., x_i, ...x_\infty\}$ be an infinite countable frame of discernment, with $x_i \cap x_j = \Phi$ for $i \neq j$, and $m_1(\cdot)$, $m_2(\cdot)$ two masses, defined as follows:

$$m_1(x_i) = a_i \in [0,1] \text{ and } m_2(x_i) = b_i \in [0,1] \text{ for all } i \in \{1, 2, ..., i, ...\infty\},$$

such that

$$\sum_{i=1}^{\infty} m_1(x_i) = 1 \text{ and } \sum_{i=1}^{\infty} m_2(x_i) = 1,$$

therefore $m_1(\cdot)$ and $m_2(\cdot)$ are normalized.

Bayesian masses.

1. Let's fusion $m_1(\cdot)$ and $m_2(\cdot)$, two Bayesian masses:

	x_1	x_2	...	x_i	...	x_j	... x_∞	Φ(*conflicting mass*)
m_1	a_1	a_2	...	a_i	...	a_j	....	
m_2	b_1	b_2	...	b_i	...	b_j	....	
m_{12}	a_1b_1	a_2b_2	...	a_ib_i	...	a_jb_j	....	$1-\sum_{i=1}^{\infty} a_ib_i$

where $m_{12}(\cdot)$ represents the conjunctive rule fusion of $m_1(\cdot)$ and $m_2(\cdot)$.

a) If we use Dempster's rule to normalize $m_{12}(\cdot)$, we need to divide each $m_{12}(x_i)$ by the sum of masses of all non-null elements, and we get:

$$m_{12DS}(x_i) = \frac{a_ib_i}{\sum_{i=1}^{\infty} a_ib_i},$$

for all i.

b) Using PCR_5 the redistribution of the conflicting mass $a_i b_j + b_i a_j$ between x_i and x_j (for all $j \neq i$) is done in the following way:

$$\frac{\alpha_i}{a_i} = \frac{\alpha_j}{b_j} = \frac{a_i b_j}{a_i + b_j}, \text{ whence } \alpha_i = \frac{a_i^2 b_j}{a_i + b_j}$$

and

$$\frac{\beta_i}{b_i} = \frac{\beta_j}{a_j} = \frac{b_i a_j}{b_i + a_j}, \text{ whence } \beta_i = \frac{b_i^2 a_j}{b_i + a_j}.$$

Therefore

$$m_{12PCR_5}(x_i) = a_i b_i + \sum_{\substack{j=1 \\ j \neq i}}^{\infty} \left(\frac{a_i^2 b_j}{a_i + b_j} + \frac{a_j b_i^2}{a_j + b_i} \right),$$

for all i.

Non-Bayesian masses.

2. Let's consider two <u>non-Bayesian masses</u> $m_3(\cdot)$ and $m_4(\cdot)$:

	x_1	x_2	...	x_i	...	x_j	... x_∞	θ	Φ(*conflicting mass*)
m_3	c_1	c_2	...	c_i	...	c_j	....	C	
m_4	d_1	d_2	...	d_i	...	d_j	....	D	
m_{34}	....			$c_i d_i + c_i D + C d_i$			....	CD	$1 - CD - \sum_{i=1}^{\infty}\left(c_i d_i + c_i D + C d_i\right)$

where $m_3(x_i) = c_i \in [0,1]$ for all i, and $m_3(\theta) = C \in [0,1]$,

and $m_4(x_i) = d_i \in [0,1]$ for all i, and $m_4(\theta) = D \in [0,1]$,

such that $m_3(\cdot)$ and $m_4(\cdot)$ are normalized:

$$C + \sum_{i=1}^{\infty} c_i = 1 \text{ and } D + \sum_{i=1}^{\infty} d_i = 1.$$

$m_{34}(x_i) = c_i d_i + c_i D + C d_i$ for all $i \in \{1, 2, ..., \infty\}$, and $m_{34}(\theta) = C \cdot D$, where $m_{34}(\cdot)$ represents the conjunctive combination rule.

a) If we use the Dempster's rule to normalize, we get:

$$m_{34DS}(x_i) = \frac{c_i d_i + c_i D + C d_i}{CD + \sum_{i=1}^{\infty}\left(c_i d_i + c_i D + C d_i\right)}$$

for all i,

and

$$m_{34DS}(\theta) = \frac{CD}{CD + \sum_{i=1}^{\infty}\left(c_i d_i + c_i D + C d_i\right)}.$$

b) If we use PCR_5, we similarly transfer the conflicting mass as in the previous 1.b) case, and we get:

$$m_{34PCR_5}(x_i) = c_i d_i + c_i D + C d_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{\infty}\left(\frac{c_i^2 d_j}{c_i + d_j} + \frac{c_j d_i^2}{c_j + d_i}\right)$$

for all i,

and $m_{34PCR_5}(\theta) = C \cdot D$

Numerical Examples.

We consider infinite positive geometrical series whose ratio $0 < r < 1$ as masses for the sets $x_1, x_2, ..., x_\infty$, so the series are congruent:

If $P_1, P_2, ..., P_n ...$ is an infinite positive geometrical series whose ratio $0 < r < 1$, then

$$\sum_{i=1}^{\infty} P_i = \frac{P_1}{1-r}$$

Example 1 (Bayesian).

Let $m_1\left(x_i\right) = \frac{1}{2^i}$ for all $i \in \{1, 2, ..., \infty\}$.

$$\sum_{i=1}^{\infty} m_1\left(x_i\right) = \sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

since the ratio of this infinite positive geometric series is $\frac{1}{2}$.

And $m_2\left(x_i\right) = \frac{2}{3^i}$ for all $i \in \{1, 2, ..., \infty\}$

$$\sum_{i=1}^{\infty} m_2\left(x_i\right) = \sum_{i=1}^{\infty} \frac{2}{3^i} = \frac{\frac{2}{3}}{1-\frac{1}{3}} = 1$$

since the ratio of this infinite positive geometric series is $\frac{1}{3}$.

	x_1	x_2	...	x_i	...	x_j	... x_∞	Φ
m_1	$\frac{1}{2}$	$\frac{1}{2^2}$	...	$\frac{1}{2^i}$	...	$\frac{1}{2^j}$	....	
m_2	$\frac{2}{3}$	$\frac{2}{3^2}$	...	$\frac{2}{3^i}$	...	$\frac{2}{3^j}$	....	
m_{12}	$\frac{2}{6}$	$\frac{2}{6^2}$	...	$\frac{2}{6^i}$	...	$\frac{2}{6^j}$	....	$1-\sum_{i=1}^{\infty}\frac{2}{6^i}=1-\frac{\frac{2}{6}}{1-\frac{1}{6}}=\frac{3}{5}$

$m_{12}(\cdot)$ is the conjunctive rule.

a) Normalizing with the Dempster's we get:

$$m_{12DS}(x_i)=\frac{\frac{2}{6^i}}{\sum_{i=1}^{\infty}\frac{2}{6^i}}=\frac{\frac{2}{6^i}}{\frac{\frac{2}{6}}{1-\frac{1}{6}}}=\frac{2}{6^i}\cdot\frac{5}{2}=\frac{5}{6^i}$$

for all i.

b) Normalizing with PCR_5 we get:

$$m_{12PCR_5}(x_i)=\frac{2}{6^i}+\sum_{\substack{j=1\\ j\neq i}}^{\infty}\left(\frac{\frac{1}{2^{2i}}\cdot\frac{2}{3^j}}{\frac{1}{2^i}+\frac{2}{3^j}}+\frac{\frac{1}{2^j}\cdot\frac{4}{6^{2i}}}{\frac{1}{2^j}+\frac{2}{3^i}}\right)$$

Example 2 (non-Bayesian).

Let $m_3(x_i)=\frac{1}{3^i}$ for all $i\in\{1,2,....,\infty\}$, and $m_3(\theta)=\frac{1}{2}$.

$$m_3(\theta)+\sum_{i=1}^{\infty}m_3(x_i)=\frac{1}{2}+\sum_{i=1}^{\infty}\frac{1}{3^i}=\frac{1}{2}+\frac{\frac{1}{3}}{1-\frac{1}{3}}=1,$$

so $m_3(\cdot)$ is normalized.

And $m_4(x_i)=\frac{1}{4^i}$ for all i, and $m_4(\theta)=\frac{2}{3}$.

$$m_4(\theta)+\sum_{i=1}^{\infty}m_4(x_i)=\frac{2}{3}+\sum_{i=1}^{\infty}\frac{1}{4^i}=\frac{2}{3}+\frac{\frac{1}{4}}{1-\frac{1}{4}}=1,$$

so $m_4(\cdot)$ is normalized.

	x_1	x_2	...	x_i	...	x_j	...	x_∞	θ	Φ
m_3	$\frac{1}{3}$	$\frac{1}{3^2}$	...	$\frac{1}{3^i}$	...	$\frac{1}{3^j}$	....		$\frac{1}{2}$	
m_4	$\frac{1}{4}$	$\frac{1}{4^2}$	...	$\frac{1}{4^i}$	...	$\frac{1}{4^j}$	....		$\frac{2}{3}$	
m_{34}	....			$\frac{1}{12^i}+\frac{2}{3^{i+1}}+\frac{1}{2\cdot 4^i}$	....				$\frac{1}{6}$	$1-\frac{1}{6}-\sum_{i=1}^{\infty}\left(\frac{1}{12^i}+\frac{2}{3^{i+1}}+\frac{1}{2\cdot 4^i}\right)=$ $=\frac{5}{6}-\frac{\frac{1}{12}}{1-\frac{1}{12}}-\frac{\frac{2}{3^2}}{1-\frac{1}{3}}-\frac{\frac{1}{2\cdot 4}}{1-\frac{1}{4}}=$ $=\frac{5}{6}-\frac{1}{11}-\frac{1}{3}-\frac{1}{6}=\frac{8}{33}$ *conflicting mass*

a) Normalizing with Dempster's rule we get:

$$m_{34DS}(x_i)=\frac{33}{25}\left(\frac{1}{12^i}+\frac{2}{3^{i+1}}+\frac{1}{2\cdot 4^i}\right)$$

for all i,
and

$$m_{34DS}(\theta)=\frac{33}{25}\cdot\frac{1}{6}=\frac{33}{150}.$$

b) Normalizing with PCR_5 we get

$$m_{34PCR_5}(x_i)=\frac{1}{12^i}+\frac{2}{3^{i+1}}+\frac{1}{2\cdot 4^i}+\sum_{\substack{j=1\\ j\neq i}}^{\infty}\left(\frac{\frac{1}{3^{2i}}\cdot\frac{1}{4^j}}{\frac{1}{3^i}+\frac{1}{4^j}}+\frac{\frac{1}{3^j}\cdot\frac{1}{4^{2i}}}{\frac{1}{3^j}+\frac{1}{4^i}}\right)$$

for all i,
and

$$m_{34PCR_5}(\theta)=\frac{1}{6}.$$

References:

1-3. F. Smarandache, J. Dezert, *Advances and Applications of DSmT for Information Fusion*, *Vols. 1-3,* AR Press, 2004, 2006, and respectively 2009.