Why should the initial and final velocities of an ideal projectile with maximum range be orthogonal?

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Abstract

While it is a well known fact that the initial and final velocity vectors of an ideal projectile, for a launching angle of 45° to the horizontal, are mutually perpendicular for maximum range; this fact is not that well known in case of other launching angles. In this article we provide three simple methods that show why the two velocities must be orthogonal. The first is based on the fact that the ranges of two projectiles launched from the same point with any speed but complimentary angles of projection have the same value of maximum range. The second deals with the total angle of rotation of the initial velocity vector. The third takes advantage of the similarity between reflection of light at parabolic mirror surfaces and the initial and final tangential velocities of the projectile path.

Key words: Projectile, maximum range, initial final velocities, orthogonal

1. Introduction

Projectile motion is an interesting topic general physics courses. One aspect of this motion that found practical application is in predicting the launching angle to maximize the range of shot put released from a point at a height h, above the ground [1]. Projectile, by definition, is an object in motion, the only force acting on which is gravity. A projectile moving in 2-D is said to be ideal, when it follows a parabolic path. The horizontal component of displacement suffered by the projectile is called the range.

2. Projectiles with launching angle of 45°

It is well known that for any speed, when the launching angle is 45° to the horizontal, the landing angle is also 45° to the horizontal for the maximum range, and the initial and final velocities $\boldsymbol{V_i}$ and $\boldsymbol{V_f}$ are orthogonal. This is also evident from the symmetry of the parabolic path about its axis. The displacement for maximum range is horizontal; the vertical component of the displacement is zero.

3. Projectiles with launching angles not equal to 45°

For angles of projection not equal to 45°, the points of launch and landing for maximum range don't lie on a level ground but lie at different levels. If the point of launch is at the ground level, point of landing lies on the other side of the peak of the path, at a height h from the ground. the launching angle θ_i is greater than 45° for this case. On the other hand, if the point of launch is from a point at a height h from the ground, the point of landing lies on the other side of the peak, at the ground level; θ_i is less than 45° for this case.

Let us consider a projectile launched from a point at a height h, from the ground with a velocity $\boldsymbol{V_i}$, at an angle θ_i (< 45°) to the horizontal, touches the ground with a velocity $\boldsymbol{V_f}$, at an angle θ_f to the horizontal. The displacement in this case has finite non-zero values for both horizontal and vertical components. In this case also, where the path of the projectile is not symmetrical about the axis, $\boldsymbol{V_i}$ and $\boldsymbol{V_f}$ are orthogonal when the range is a maximum. The question naturally arises as to why it is so. This question was posed and discussed by Bose earlier [2, 3]. We give some more and simpler methods showing why the two velocity vectors must be orthogonal.

(i) Method based on properties of parabola

(ii) Method employing the similarity with reflection of light by parabolic mirrors

(iii) Method employing the total angle of rotation of the initial velocity vector

We will examine these methods in detail below.

(i) Method based on Properties of parabola

Two projectiles launched from a point O on a level ground with the same speed V_i but different angles of projection θ_i have different values of range $R(\theta_i)$ [4]. However, an interesting point about these projectiles is that, pairs of projectiles with complimentary angles of projection have equal range. For example, pairs with 30° and 60°; the pair with 15° and 75° and in general a pair with θ_i and θ_f each have the same range. The maximum range $R(\theta_i)$ of a projectile with a launching angle θ_i is given by,

$$R(\theta_i) = \frac{V_i^2 sin(2\theta_i)}{g} = \frac{V_i^2 sin(180 - 2\theta_i)}{g} = \frac{V_i^2 sin2(90 - \theta_i)}{g} = \frac{V_i^2 sin(2\theta_f)}{g} \qquad (1)$$

We can now get a very interesting result that is applicable to cases where the point of launch and landing are at different levels, by writing $\theta_i = (45 - \alpha)$. Then we get $\theta_f = (45 + \alpha)$ and $(\theta_i + \theta_f) = 90$. Thus θ_i and θ_f are complimentary angels and the corresponding velocity vectors are orthogonal.

Let us consider a projectile (Fig.1) launched from a point P at a height h, above a level ground with an

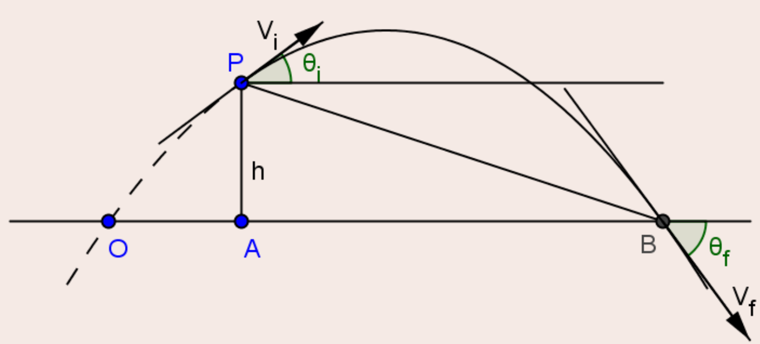

Fig. 1 Path of an ideal projectile launched from a point P at a height h from a level ground with a velocity V_i inclined at an angle of θ_i to the horizontal landing at point B with a final velocity V_f inclined at an angle of θ_f to the horizontal. AB is the horizontal component of the displacement PB. The dashed line shows the remaining part of the parabolic path.

initial velocity $\boldsymbol{V_i}$ making an angle θ_i ($0 < \theta_i < 45$) with the horizontal and landing at a point B on the ground with a final velocity $\boldsymbol{V_f}$ making an angle θ_f.

This projectile has the same range as that of a second projectile launched from B with an initial velocity '-$\boldsymbol{V_f}$' making an angle ($180 - \theta_f$) with the horizontal and landing at the point O on a level ground with a final velocity $\boldsymbol{V_f}$ making an angle ($180 + \theta_f$). B, P and O are on the same parabola, as can be seen from Fig. 2a.

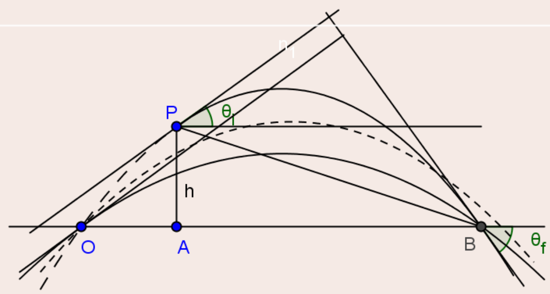

Fig. 2a The parabolic curve PB corresponds to the first projectile. The parabolic curve BPO corresponds to the second projectile. The parabolic curve OB corresponds to the third projectile. θ_i and θ_f are the angles of velocity vectors of launching and landing. The second and the first projectile have the same range. θ_i. The parabolic in dashed line corresponds to launching angle of 45°.

Again, a third projectile launched from point O with an initial velocity $\boldsymbol{V'_i}$, having the same magnitude of $\boldsymbol{V_f}$ but making an angle θ_i with the horizontal has the same range as that of the second projectile. Two projectiles launched from the same point on a level ground with the same speed but complimentary angles of projection θ_i and θ_f land at the same point have the same range as seen from eqs (1) and Fig. 2a.

Therefore, it follows that $\boldsymbol{V_i}$ and $\boldsymbol{V_f}$ of the first projectile are such that two projectiles launched from the same point on a level ground with velocities $\boldsymbol{V_i}$ and $\boldsymbol{V_f}$ having equal speeds but different angles of projection of θ_i and θ_f have the same range. The equality of the ranges demands that the two launch angles be complimentary. This is another way of saying that the initial and final velocities $\boldsymbol{V_i}$ and $\boldsymbol{V_f}$ of the first projectile must be orthogonal.

That $\boldsymbol{V_i}$ and $\boldsymbol{V_f}$ must be orthogonal may also be seen more clearly by reflecting the path of the third projectile in the line OB as shown in Fig. 2b. The directions of the tangents at the point B to the paths of the first projectile and the reflected path of the third projectile are same as the directions of $\boldsymbol{V_f}$ and $\boldsymbol{V_i}$. Since the two tangents are orthogonal, $\boldsymbol{V_i}$ and $\boldsymbol{V_f}$ must also be orthogonal.

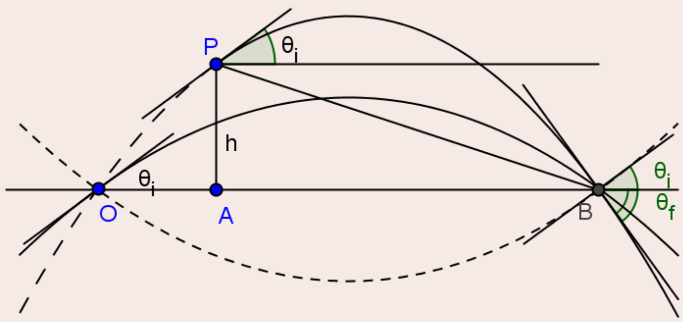

Fig. 2b. The parabola that is open upwards is the reflection of the pathof the third projectile. The tangents at B to the first parabola and the reflected parabola which correspond to the directions of V_i and V_f are orthogonal, so are V_i and V_f.

(ii) Method employing total angle of rotation of the initial velocity vector of the projectile

A projectile launched at 45° from a point on the ground has the maximum range. A projectile launched at an angle lower than 45° from a point above the ground has the maximum range when it reaches a point on

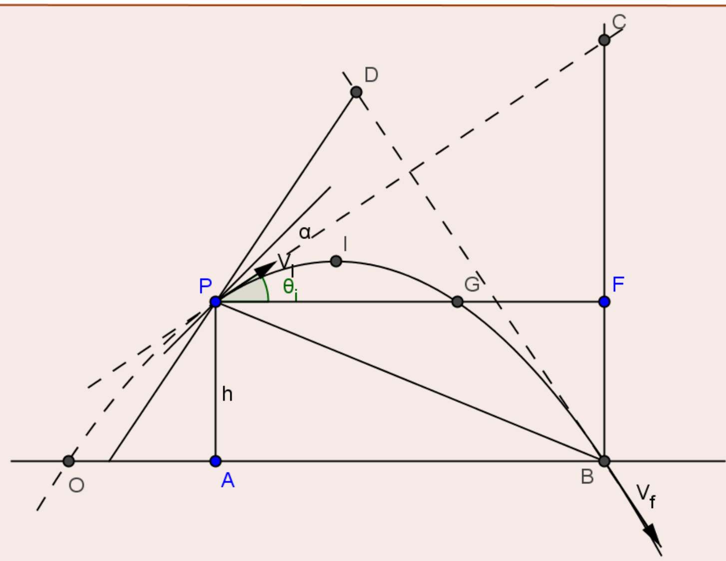

Fig. 3. The initial velocity vector of the projectile rotates as it moves from the point of launch P to the point of landing B through an angle of ($2\theta_i$ + 2α) and changes to V_f at B. The bisector of the angle CPD (2α) is parallel to the tangent of the path of the projectile launched at 45° from a point on the ground. Obviously, θ_i = (45 - α).

the ground. Similarly, a projectile launched at an angle greater than 45° from a point on the ground has the maximum range when it reaches a certain point above the ground. When a projectile is launched, its initial velocity vector keeps changing in both magnitude and direction as it moves along the parabolic path. Keeping this in mind let us try to find a solution to our problem.

Let us consider a projectile launched at an angle less than 45°, say $(45 - \alpha) = \theta_i$ from a point P, at a height h, above the ground. The velocity vector $\boldsymbol{V_i}$ keeps on rotating due to the action of the force of gravity on the projectile. It rotates through an angle of θ_i when it reaches the maximum height, and rotates through another θ_i when it reaches symmetry point of P on its path. From there, it rotates through an angle α when it reaches an inclination of -45°. It then rotates through another α degrees, when it finally touches the ground at an inclination of $-(45 + \alpha) = -\theta_f$. Thus the total rotation is through an angle of $(2\theta_i + 2\alpha)$. This gives the angle of inclination of the final velocity vector $\boldsymbol{V_f}$ to be $(\theta_i - (2\theta_i + 2\alpha)) = -\theta_f$. Since $\theta_i = (45 - \alpha)$, we get $(\theta_i + \theta_f) = 90°$. Therefore, the initial and final velocity vectors are mutually orthogonal.

(iii) Behavior of light gives another reason for the velocity vectors to be orthogonal.

When a ray of light parallel to the axis of a parabolic mirror strikes a point on the reflecting surface, it passes through the focus of the parabola (Fig. 4). Tangents to the parabola drawn from the end points of the focal chord intersect at right angles at a point on the directrix of the parabola. Since motion in nature chooses the most economical path of travel by way of distance and time, the displacement suffered by the projectile must also correspond to the displacement suffered by light between the points on the parabola. This demands that the initial and final velocity vectors of an ideal projectile for maximum range must be orthogonal.

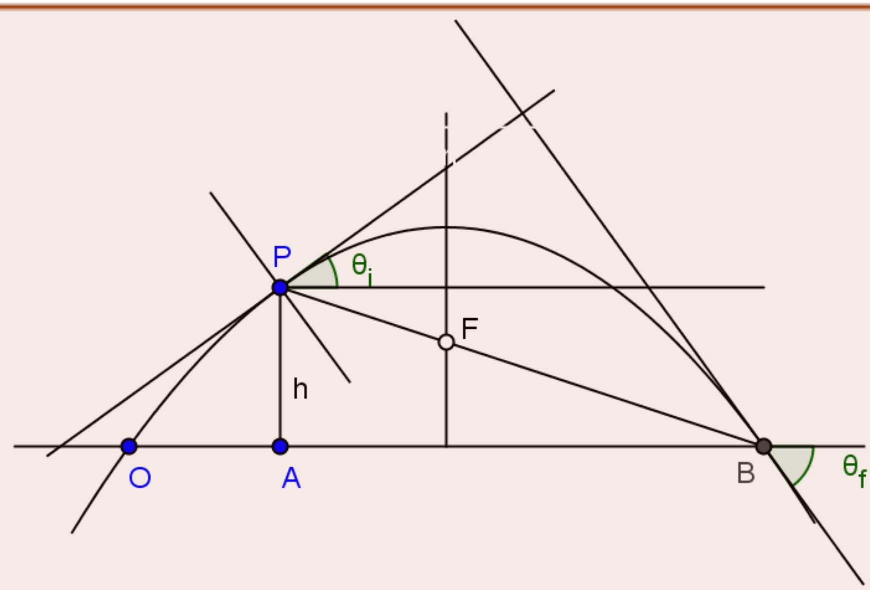

Fig. 4. Light ray AP parallel to the axis of the parabolic mirror OPBstrikes the mirror at P and after reflection passes through the focus F, to the point B on the mirror. Tangents at the end points of the focal chord PB intersect at 90° at a point on the directrix.

Fig. 5 depicts the triangle of velocities given by Bose [2], superposed on the diagram of the path of a projectile corresponding to the case of $\theta_i < 45°$.

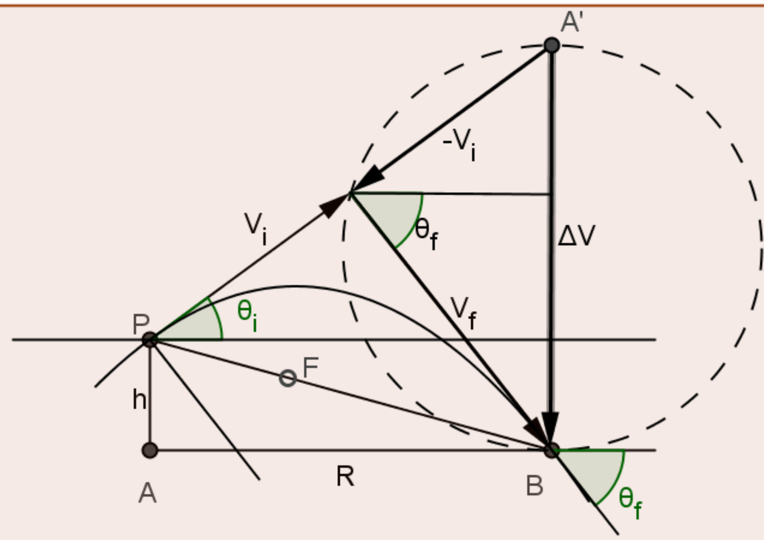

Fig. 5. A projectile is launched from point P, h units above a level ground in a uniform gravitational field, with a velocity V_i inclined at an angle θ_i to the horizontal. It follows a parabolic path and touches the ground at B with a velocity V_f inclined at an angle θ_f to the horizontal. AB corresponds to the maximum range. ΔV is the change in velocity suffered by the projectile.

4. References

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