On The Leibniz Rule And Fractional Derivative For Differentiable And Non-Differentiable Functions

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Abstract

In the recent paper *Communications in Nonlinear Science and Numerical Simulation. Vol.18. No.11. (2013) 2945-2948*, it was demonstrated that a violation of the Leibniz rule is a characteristic property of derivatives of non-integer orders. It was proved that all fractional derivatives $\mathcal{D}^\alpha$, which satisfy the Leibniz rule $\mathcal{D}^\alpha(fg) = (\mathcal{D}^\alpha f)\, g + f\, (\mathcal{D}^\alpha g)$, should have the integer order $\alpha = 1$, i.e. fractional derivatives of non-integer orders cannot satisfy the Leibniz rule. However, it should be noted that this result is only for differentiable functions. We argue that the very reason for introducing fractional derivative is to study non-differentiable functions. In this note, we try to clarify and summarize the Leibniz rule for both differentiable and non-differentiable functions. The Leibniz rule holds for differentiable functions with classical integer order derivative. Similarly the Leibniz rule still holds for non-differentiable functions with a concise and essentially local definition of fractional derivative. This could give a more unified picture and understanding for Leibniz rule and the geometrical interpretation for both integer order and fractional derivative.

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1 Introduction

One longstanding problem of fractional calculus is that there exists too many definitions [2, 4, 1, 12, 15, 10, 5, 17, 11, 20, 3, 13, 18, 14, 16] while lacking physical or geometrical meanings.

There are different definitions of fractional derivatives such as Riemann-Liouville, Riesz, Caputo, Grünwald-Letnikov, Marchaud, Weyl, Sonin-Letnikov and others [17]. Unfortunately most these fractional derivatives have a lot of unusual properties. The well-known Leibniz rule $\mathcal{D}^\alpha(fg) = (\mathcal{D}^\alpha f)g + f(\mathcal{D}^\alpha g)$ is not satisfied for differentiation of non-integer orders [17].

For example, we have the infinite series

$$\mathcal{D}^\alpha(fg) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} (\mathcal{D}^{\alpha-k} f)\,(D^k g) \tag{1}$$

for analytic functions on $[a, b]$ (see Theorem 15.1 in [17]), where $\mathcal{D}^\alpha$ is the Riemann-Liouville derivative, D^k is derivative of integer order k. Note that the sum is infinite and contains integrals of fractional order for $k > [\alpha] + 1$.

Given all the unusual properties, there are some attempts to define new type of fractional derivative such that the Leibniz rule holds (for example, see [7, 9, 8, 6, 22, 21]).

For all these different definitions about fractional derivative, each has some advantages and disadvantages. The following are some very important and fundamental questions for studying fractional calculus:

Geometric Interpretation A well-defined derivative should have reasonable geometric interpretation, like the classical one.

Local It should be local in nature, thus does not rely on information of domain and boundary conditions.

Derivative of Constant The derivative of constant function should be zero.

Generality It should be applicable to a large class of functions (i.e. continuous but non-differentiable) which are not differentiable in the classical sense.

Calculation The calculation should be easy and straightforward.

Some different definitions are reviewed and some key issues are summarized and clarified in [21]. Here this paper focus on the Leibniz rule.

2 "No violation of the Leibniz rule. No fractional derivative"

In the recent paper [19], it was demonstrated that a violation of the Leibniz rule is a characteristic property of derivatives of non-integer orders. It was proved that all fractional derivatives $\mathcal{D}^\alpha$, which satisfy the Leibniz rule $\mathcal{D}^\alpha(fg) = (\mathcal{D}^\alpha f)\, g + f\,(\mathcal{D}^\alpha g)$, should have the integer order $\alpha = 1$, i.e. fractional derivatives of non-integer orders cannot satisfy the Leibniz rule.

In [19] fractional derivatives $\mathcal{D}^\alpha$ of non-integer orders α was considered by using an algebraic approach. Special forms of fractional derivatives are not important for our consideration.

For the operator $\mathcal{D}^\alpha$, the following conditions are considered.

1) $\mathbb{R}$-linearity:

$$\mathcal{D}^\alpha_x (c_1 f(x) + c_2 g(x)) = c_1 (\mathcal{D}^\alpha_x f(x)) + c_2\, (\mathcal{D}^\alpha_x g(x)), \tag{2}$$

where c_1 and c_2 are real numbers.

2) The Leibniz rule:

$$\mathcal{D}^\alpha_x (f(x)\, g(x)) = (\mathcal{D}^\alpha_x f(x))\, g(x) + f(x)\, (\mathcal{D}^\alpha_x g(x)). \tag{3}$$

3) If the linear operator satisfies the Leibniz rule, then the action on the unit (and on a constant function) is equal to zero:

$$\mathcal{D}^\alpha_x 1 = 0. \tag{4}$$

Theorem 1 ("No violation of the Leibniz rule. No fractional derivative"). *[19]*
If an operator $\mathcal{D}^\alpha_x$ can be applied to functions from $C^2(U)$, where $U \subset \mathbb{R}^1$ be a neighborhood of the point x_0, and conditions (2), (3) are satisfied, then the operator $\mathcal{D}^\alpha_x$ is the derivative D^1_x of integer (first) order, i.e. it can be represented in the form

$$\mathcal{D}^\alpha_x = a(x)\, D^1_x, \tag{5}$$

where $a(x)$ are functions on $\mathbb{R}^1$.

However, it should be noted that this result is only for differentiable functions, applied to functions from $C^2(U)$. We argue that the very reason for introducing fractional derivative is to study non-differentiable functions. In this note, we try to clarify and summarize the Leibniz rule for both differentiable and non-differentiable functions. The Leibniz rule holds for differentiable functions with classical integer order derivative. Also the Leibniz rule still holds for non-differentiable functions with a concise and essentially local definition of fractional derivative. This could give a more unified picture and understanding for Leibniz rule and the geometrical interpretation for both integer order and fractional derivative.

3 A simple definition directly from geometrical meaning

We expect that **the fractional derivative could give nonlinear (power law) approximation of the local behavior of non-differentiable functions**:

$$f(x+h) \approx f(x) + \frac{f^{(\alpha)}(x)}{\Gamma(1+\alpha)} h^\alpha \tag{6}$$

in which the function f is not differentiable because $df \approx (dx)^\alpha$ so the classical derivative df/dx will diverge. Note that the purpose of adding the coefficient $\Gamma(1+\alpha)$ is just to make the formal consistency with the Taylor series.

We can give a very simple definition directly from the above meaning:

Definition 1. *For function $f \in C^\alpha,\ 0 < \alpha < 1$, the fractional derivative is defined as*

$$f^{(\alpha)}(x) := \Gamma(1+\alpha) \lim_{h \to 0+} \frac{f(x+h) - f(x)}{h^\alpha} \tag{7}$$

And we call $f(x)$ is α-differentiable on x_0 if $f^{(\alpha)}(x_0)$ exist. We call $f(x)$ is exactly α-differentiable on x_0 if $f(x+h) - f(x)$ and h^α is of the same order.

Remark 1.

1. *The derivative of constant function is zero.*
2. *It could be applicable to a large class of functions C^α $0 < \alpha < 1$ which are not differentiable in the classical sense.*
3. *It do have reasonable geometric interpretation, similar to the classical one. The fractional derivative could give nonlinear (power law) approximation of the local behavior of non-differentiable functions.*
4. *The calculation of derivative is numerically easy.*
5. *Last but not least, this definition is* local *by nature. This property facilitate the generalization of one variable fractional derivative to vector derivative, which is very hard if the nonlocal domain and boundary condition of the function is needed for calculation.*

The local behavior of a non-differentiable function is quite different from any smooth function like $x^n, e^x, \sin x$. So it's not trivial to find suitable non-differentiable function in expression by fundamental smooth function like $x^n, e^x, \sin x$. Here we can give such example expressed by fundamental smooth function, but these examples are only non-differentiable on a single point.

Example 1. x^β *for* $0 < \beta < 1$
$u(x) = x^\beta$ is smooth when $x > 0$, it's not differentiable only at $x = 0$. So for $x > 0$ the $\mathcal{D}^\alpha x^\beta, \alpha < 1$ is trivially vanished. The only interesting results can be found at $x = 0$.

$$\mathcal{D}^\alpha x^\beta|_{x=0} = \Gamma(1+\alpha) \lim_{h \to 0+} \frac{(0+h)^\beta - 0^\beta}{h^\alpha} = \begin{cases} 0, & 0 < \alpha < \beta < 1 \\ \Gamma(1+\alpha), & \alpha = \beta \\ \infty, & 0 < \beta < \alpha < 1 \end{cases} \tag{8}$$

Example 2. *Mittag-Leffler function*
Another interesting example is the Mittag-Leffler function $v(x) = E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+\alpha k)}$ *at* $x = 0$ *which also have non-trivial* α*-derivative.*

$$\mathcal{D}^\alpha E_\alpha(x^\alpha)|_{x=0} = \mathcal{D}^\alpha \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+\alpha k)}|_{x=0} = \Gamma(1+\alpha) \lim_{h\to 0+} \frac{\sum_{k=1}^{\infty} \frac{h^{\alpha k}}{\Gamma(1+\alpha k)}}{h^\alpha} = E_\alpha(x^\alpha)|_{x=0} = 1 \quad (9)$$

4 Leibniz rule still holds for non-differentiable functions

It follows directly from the definition of α-derivative that if j and k are both α-differentiable on x_0, then the Leibniz rule still holds (for non-differentiable functions in classical sense while α-differentiable in the new sense):

$$\begin{aligned}
f^{(\alpha)}(x_0) &= \Gamma(1+\alpha) \lim_{h\to 0+} \frac{f(x_0+h) - f(x_0)}{h^\alpha} \\
&= \Gamma(1+\alpha) \lim_{h\to 0+} \frac{j(x_0+h)\,k(x_0+h) - j(x_0)\,k(x_0)}{h^\alpha} \\
&= \Gamma(1+\alpha) \lim_{h\to 0+} \frac{j(x_0+h)\,k(x_0+h) - j(x_0+h)\,k(x_0) + j(x_0+h)\,k(x_0) - j(x_0)\,k(x_0)}{h^\alpha} \\
&= \Gamma(1+\alpha) \lim_{h\to 0+} \left(j(x_0+h) \frac{k(x_0+h) - k(x_0)}{h^\alpha} + \frac{j(x_0+h) - j(x_0)}{h^\alpha} k(x_0) \right) \\
&= j(x_0)\; k^{(\alpha)}(x_0) + j^{(\alpha)}(x_0)\; k(x_0)
\end{aligned}$$

Example 3. α*-derivative Leibniz rule*
As shown in above examples, non-differentiable functions in classical sense could still be α*-differentiable in the new sense. Both* x^α *and* $E_\alpha(x^\alpha)$ *are* α*-differentiable at* $x = 0$*.*

$$\begin{aligned}
\mathcal{D}^\alpha[u(x)v(x)]|_{x=0} &= \mathcal{D}^\alpha[x^\alpha E_\alpha(x^\alpha)]|_{x=0} \\
&= \mathcal{D}^\alpha \sum_{k=0}^{\infty} \frac{x^{\alpha(k+1)}}{\Gamma(1+\alpha k)}|_{x=0} \\
&= \Gamma(1+\alpha) \lim_{h\to 0+} \frac{\sum_{k=0}^{\infty} \frac{h^{\alpha(k+1)}}{\Gamma(1+\alpha k)}}{h^\alpha}|_{x=0} = \Gamma(1+\alpha)
\end{aligned}$$

At the other hand,

$$\begin{gathered}
[\mathcal{D}^\alpha u(x)]v(x)|_{x=0} + [\mathcal{D}^\alpha v(x)]u(x)|_{x=0} \\
= [\Gamma(1+\alpha) + x^\alpha] E_\alpha(x^\alpha)|_{x=0} = \Gamma(1+\alpha)
\end{gathered}$$

To conclude, the Leibniz rule holds for 1-differentiable functions with classical first order derivative, similarly the Leibniz rule still holds for α-differentiable functions with α-derivative. This could give a more unified picture and understanding for Leibniz rule and the geometrical interpretation for both integer order and fractional derivative.

References

[1] Ronald L Bagley and J Torvik, P. Fractional calculus-a different approach to the analysis of viscoelastically damped structures. *AIAA journal*, 21(5):741–748, 1983.

[2] Ronald L Bagley and Peter J Torvik. On the fractional calculus model of viscoelastic behavior. *Journal of Rheology (1978-present)*, 30(1):133–155, 1986.

[3] Dumitru Baleanu. *Fractional Calculus: Models and Numerical Methods*, volume 3. World Scientific, 2012.

[4] Shantanu Das. *Functional fractional calculus*. Springer, 2011.

[5] Kai Diethelm and Neville J Ford. Analysis of fractional differential equations. *Journal of Mathematical Analysis and Applications*, 265(2):229–248, 2002.

[6] Guy Jumarie. Stochastic differential equations with fractional brownian motion input. *International journal of systems science*, 24(6):1113–1131, 1993.

[7] Guy Jumarie. Modified riemann-liouville derivative and fractional taylor series of non-differentiable functions further results. *Computers & Mathematics with Applications*, 51(9):1367–1376, 2006.

[8] Guy Jumarie. Lagrangian mechanics of fractional order, hamilton–jacobi fractional pde and taylors series of nondifferentiable functions. *Chaos, Solitons & Fractals*, 32(3):969–987, 2007.

[9] Guy Jumarie. Table of some basic fractional calculus formulae derived from a modified riemann–liouville derivative for non-differentiable functions. *Applied Mathematics Letters*, 22(3):378–385, 2009.

[10] RC Koeller. Applications of fractional calculus to the theory of viscoelasticity. *Journal of Applied Mechanics*, 51(2):299–307, 1984.

[11] Kiran M Kolwankar and Anil D Gangal. Fractional differentiability of nowhere differentiable functions and dimensions. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 6(4):505–513, 1996.

[12] Francesco Mainardi. Fractional calculus: some basic problems in continuum and statistical mechanics. *arXiv preprint arXiv:1201.0863*, 2012.

[13] Agnieszka B Malinowska and D Torres. *Fractional Calculus of Variations*. Imperial College Press, Singapore, 2012.

[14] Kenneth S Miller and Bertram Ross. An introduction to the fractional calculus and fractional differential equations. 1993.

[15] Bertram Ross. *Fractional calculus and its applications*. Springer Berlin, 1975.

[16] J Sabatier, Om P Agrawal, and JA Tenreiro Machado. *Advances in fractional calculus*. Springer, 2007.

[17] Stefan G Samko, Anatolij A Kilbas, and Oleg I Maričev. *Fractional integrals and derivatives*. Gordon and Breach Science Publ. Yverdon, 1993.

[18] Enrico Scalas, Rudolf Gorenflo, and Francesco Mainardi. Fractional calculus and continuous-time finance. *Physica A: Statistical Mechanics and its Applications*, 284(1):376–384, 2000.

[19] Vasily E Tarasov. No violation of the leibniz rule. no fractional derivative. *Communications in Nonlinear Science and Numerical Simulation*, 18(11):2945–2948, 2013.

[20] Vladimir V Uchaikin, Vladimir V Uchaikin, and Vladimir V Uchaikin. *Fractional derivatives for physicists and engineers*. Springer, 2013.

[21] Xiong Wang. Fractional geometric calculus: toward a unified mathematical language for physics and engineering. In *Proceedings of the Fifth Symposium on Fractional Differentiation and Its Applications (FDA2012), Hohai University, Nanjing*, 2012.

[22] Xiao-Jun Yang. Advanced local fractional calculus and its applications. *World Science, New York, NY, USA*, 2012.