New solutions of the Tolman-Oppenheimer-Volkov equation and of Kerr space-time with matter and the corresponding star models

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Abstract
The Tolman-Oppenheimer-Volkov (TOV) equation is solved with a new ansatz: the external boundary condition with mass M_0 and radius R_1 is dual to the internal boundary condition with density ρ_{bc} and inner radius r_i , the two boundary conditions yield the same result. The inner boundary condition is imposed with a density ρ_{bc} and an inner radius r_i, which is zero for the compact neutron stars, but non-zero for the shell-stars: stellar shell-star and galactic (supermassive) shell-star. Parametric solutions are calculated for neutron stars, stellar shell-stars, galactic shell-stars. From the results an M-R-relation and mass limits for these star models can be extracted. A new method is found for solving the Einstein equations for Kerr space-time with matter (extended Kerr space-time), i.e. rotating matter distribution in its own gravitational field. Then numerical solutions are calculated for several astrophysical models: white dwarf, neutron star, stellar shell-star, galactic shell-star. The results are that shell-star star models closely resemble the behaviour of abstract black holes, including the Bekenstein-Hawking entropy, but have finite redshifts and escape velocity v<c and no singularity .

1. Introduction
In General Relativity, one of the most important applications is to calculate the mass distribution and the space-time metric for a given equation-of-state of a stellar model. Without rotation, one has spherical symmetry and in then the Tolman-Oppenheimer-Volkov (TOV) equation in radius *r*, which is derived directly from the Einstein equations (see [2]) , is being used. The TOV equation consists originally of 2 coupled non-linear ordinary differential equations (odeq) of degree 1 in *r* for mass *M(r)* and density $\rho(r)$, where $4\pi r^2 \rho(r) = M(r)$, and can be transformed into one ordinary differential equation of degree 2 for *M(r)* by eliminating $\rho(r)$.
The boundary condition is imposed normally at *r=0* with *M(0)=0* and $\rho(0)=\rho_0$, where ρ_0 is the maximal density. Then the TOV-equation for *M(r)* is solved with this boundary condition at *r=0* for *M(r)* and *M'(r)* , which gives the total mass $M0(\rho_0)$ and the total radius $R(\rho_0)$, and a *mass-radius relation M0(R)* .
The predominant view of the neutron stars and stellar black-holes is, that neutron stars obey an equation-of state (eos)of an interacting-fluid model [6], which solutions of the TOV equation up to about M=3 M_{sun} .For larger masses, so it is assumed, only a black-hole solution remains. This is based on the so-called Oppenheimer limit for the radius of a compact mass $R_{\lim} = \frac{9}{8} r_s = \frac{9MG}{4c^2}$: for a smaller compact object, the density at the center becomes infinite.
The new ansatz presented here is the extended (inner) boundary condition at $r=r_i$ with the *non-zero inner radius* r_i , $M(r_i)=0$ and $\rho(r_i)=\rho_0$, i.e. the star becomes a *shell-star* with an

(almost) void interior. With the parameters r_i and ρ_0 this ansatz generates a 2-parametric solution manifold, where, because of energy minimization, the stable physical solution is the one with minimal r_i for a given ρ_0, which determines the total mass $M0(r_i, \rho_0)$ and the total radius $R(r_i, \rho_0)$. This ansatz circumvents the Oppenheimer limit, because the mass is non-compact, and the density at the center is zero. It yields valid solutions of the TOV-equation with *a continuous mass-radius relation* $M0(R, r_i)$.
The dual (outer) boundary condition is the one at $r=R$ with $M=M_0$ and $\rho=\rho_{bc}$, where ρ_{bc} depends on the equation-of-state (eos): for neutron stars with interacting nucleon fluid $\rho_{bc} = \rho_c > 0$ with the equilibrium nucleon density ρ_c, and for the eos of non-interacting nucleon Fermi-gas (stellar shell-stars) $\rho_{bc} = 0$. The 2 parameters R and M0 in the dual outer boundary condition correspond uniquely to the 2 parameters r_i and ρ_0 in the inner boundary condition.
With rotation, one has an axisymmetric model in the variables r and θ (azimuthal angle), and has to solve the Einstein equations in these 2 coordinates. In vacuum, the corresponding solution is the Kerr space-time in r and θ. With mass, a good starting point is using the *extended* Kerr space-time in Boyer-Lindquist coordinates with correction-factor functions $A0,\ldots,A4$ and $B0,\ldots B4$ and the mass $M(r, \theta)$ as variables and insert this into the Einstein equations. Setting $B_i=0$ some of the 10 Einstein equations become trivial and one is left with 6 partial-differential equations (pdeq) in r and θ for the 6 variables $A0,\ldots,A4$ and M.
The (outer) boundary condition here at the effective star radius R with total mass M_0 is: $A_i=1$, $M=M_0$ and $\partial_r A_i=0$, $\partial_r M=0$, as the density becomes 0 and the space-time becomes the normal Kerr space-time in vacuum.
Now, with rotation, we have a new model parameter, the angular velocity ω, to which corresponds a third parameter in the outer boundary condition: (outer) ellipticity ΔR_1, where $R_{1x}=R_{1y}-\Delta R_1$ and R_{1x} and R_{1y} are the equatorial and the polar radius. As in the TOV-case, here to the 3 parameters R_{1y}, M_0 and ΔR_1 correspond the 3 inner parameters r_{iy}, ρ_0 and Δr_i. So here we get a 3-parametric solution manifold, and as in the spherical case, for a given total mass M0 we have to find the stable physical solution. As before, these will be the ones with minimal r_{iy} and among them the one with minimal mean energy density: this defines the inner ellipticity Δr_i. In all considered cases, it can be shown numerically, that such a (non-trivial) minimum exists.
The paper is organized as follows.
In 2 we present the mathematical setup, in 3 the equations for the extended Kerr space-time with rotation, in 4 the solution algorithm for it. In 5 the TOV-equation is introduced, in 6 the equation-of-state for the nucleon fluid and nucleon gas. In 7 the results for the TOV-equation are shown: the parametric solution manifold in 7.1. and the case study for typical stars in 7.2. In 8 the results for the extended Kerr space-time with rotation are presented for three typical star configurations: compact neutron star, stellar shell-star, galactic shell-star.

2. The Kerr space-time, Schwarzschild space-time, Einstein equations

Using the Minkowski metric $\eta_{\mu\nu}$ = diag(1,-1,-1,-1), the Kerr space-time metric in original Kerr coordinates (u, θ, ϕ) has the line element [2]

$$ds^2 = \left(1 - \frac{r r_s}{r^2 + a^2\cos^2\theta}\right)\left(du + a\sin^2\theta\, d\varphi\right)^2 - 2\left(du + a\sin^2\theta\, d\varphi\right)\left(dr + a\sin^2\theta\, d\varphi\right) - \left(r^2 + a^2\cos^2\theta\right)\left(d\theta^2 + \sin^2\theta\, d\varphi\right) \quad (1)$$

where $r_s = \frac{2GM}{c^2}$ is the Schwarzschild radius, and $a = \frac{J}{Mc}$ is the angular momentum radius (amr) , *a* has the dimension of a distance: $[a] = [r]$, and *J* is the angular momentum. With this line element the Kerr metric tensor $g_{\mu\nu}$ is as follows:

$$g_{\mu\nu} = \begin{pmatrix} \left(1-\frac{rr_s}{\rho_{12}}\right) & -1 & 0 & -\frac{rr_s a\sin^2\theta}{\rho_{12}} \\ & 0 & 0 & -a\sin^2\theta \\ & & -\rho_{12} & 0 \\ & & & g_{33} \end{pmatrix} \qquad (2)$$

with the abbreviations $\rho_{12} = (r^2 + a^2\cos^2\theta)$

and $g_{33} = -\sin^2\theta(r^2 + a^2 + \frac{rr_s a^2 \sin^2\theta}{\rho_{12}})$.

In the limit a->0 the Schwarzschild space-time in advanced Eddington-Finkelstein coordinates emerges:

$$ds^2 = \left(1-\frac{r_s}{r}\right)du^2 - 2du\,dr \; - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi\right) \qquad (3)$$

This form of the Schwarzschild line element has the advantage in comparison with the original line element

$$ds^2 = \left(1-\frac{r_s}{r}\right)c^2dt^2 - \frac{dr^2}{\left(1-\frac{r_s}{r}\right)} \; - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi\right) \qquad (3a)$$

that the (apparent) singularity at $r = r_s$ is missing .

The same is valid for the original Kerr space-time: the denominator ρ_{12} has no zeros, there is no singularity in g_{ab}, which makes it more well-behaved numerically.

Alternatively, in Boyer-Lindquist-coordinates:

$$g_{\mu\nu} = \begin{pmatrix} \left(1-\frac{rr_s}{\rho_{12}}\right) & 0 & 0 & \frac{rr_s a\sin^2\theta}{\rho_{12}} \\ & -\rho_{12}/\Lambda_{12} & 0 & 0 \\ & & -\rho_{12} & 0 \\ & & & -\sin^2\theta(r^2 + a^2 + \frac{rr_s a^2\sin^2\theta}{\rho_{12}}) \end{pmatrix} \qquad (4)$$

with the line element

$$ds^2 = \left(1-\frac{rr_s}{r^2+a^2\cos^2\theta}\right)(dt)^2 \; + \left(\frac{2rr_s a\sin^2\theta}{r^2+a^2\cos^2\theta}\right)dt\,d\varphi$$
$$-\left(\frac{r^2+a^2\cos^2\theta}{r^2-rr_s+a^2}\right)dr^2 -$$
$$\left(r^2+a^2+\frac{rr_s a^2\sin^2\theta}{r^2+a^2\cos^2\theta}\right)\sin^2\theta\, d\varphi^2 - \left(r^2+a^2\cos^2\theta\right)\left(d\theta^2\right) \qquad (1a)$$

with the abbreviation $\Lambda_{12} = r^2 - r r_s + a^2$. Here, Λ_{12} has zeros at the inner/outer horizon $r = (r_s/2) \pm \sqrt{(r_s/2)^2 - a^2 \cos^2\theta}$, so for numerical calculations the singularity has to be removed by adding a small ε : $\Lambda s_{12} = \sqrt{(r^2 - r r_s + a^2)^2 + \varepsilon^2}$.
In the limit a->0 the Schwarzschild space-time in the standard form (4) emerges.

The Einstein field equations with the above Minkowski metric are:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R_0 - \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu} \tag{5}$$

where $R_{\mu\nu}$ is the Ricci tensor, R_0 the Ricci curvature, $\kappa = \frac{8\pi G}{c^4}$, $T_{\mu\nu}$ is the energy-momentum tensor, Λ is the cosmological constant (in the following neglected, i.e. set 0), with the Christoffel symbols (second kind)

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} \left(\frac{\partial g_{\kappa\mu}}{\partial x^{\nu}} + \frac{\partial g_{\kappa\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\kappa}} \right) \tag{6}$$

and the Ricci tensor

$$R_{\mu\nu} = \frac{\partial \Gamma^{\rho}_{\mu\rho}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\rho}} + \Gamma^{\sigma}_{\mu\rho} \Gamma^{\rho}_{\sigma\nu} - \Gamma^{\sigma}_{\mu\nu} \Gamma^{\rho}_{\sigma\rho} \tag{7}$$

The crucial part of the extended Kerr solution is the expression for the energy-momentum tensor $T_{\mu\nu}$. As usual, one uses the formula for the perfect fluid [2,(45.3)]:

$$T_{\mu\nu} = \left(\rho + \frac{P}{c^2} \right) u_{\mu} u_{\nu} - P g_{\mu\nu} \tag{8}$$

where *P* and ρ is the pressure and density, u_{μ} is the covariant velocity 4-vector.
In the Schwarzschild case, when deriving the TOV-equation, one sets the spatial contravariant velocity components to 0: $u^i = 0$, in the Kerr case the tangential velocity $u^3 = u^{\varphi} \neq 0$.
For the velocity one has:
$u^3 = r\,\omega = r \frac{a\,M\,c}{I}$, where *I* is the moment of inertia and ω the angular velocity,
$I = f_I M R_1^{\,2}$, so

$$u^3 = \frac{1}{f_I} \frac{a\ cr}{R_1^{\,2}}$$

for a homogeneous sphere $I = \frac{2}{5} MR^2$ i.e. $f_I = \frac{2}{5}$, for a thin shell $I = \frac{2}{3} MR^2$ i.e. $f_I = \frac{2}{3}$, for a disc $I = \frac{1}{2} MR^2$.
If we make the obvious assumption that the star rotates as a whole, i.e. with constant angular velocity, then the moment of inertia *I* becomes r-dependent, like the mass *M* :

$$M(r) = \int_0^r \rho(r_1) 3 r_1^{\,2}\, dr_1 \tag{9}$$

$$I(r) = \int_0^r \rho(r_1) 3 r_1^{\,4}\, dr_1$$

The factor 3 in the integral instead of the usual 4π comes from the dimensionless calculation in “sun units” (see below).
The amr *a* also becomes r-dependent:

$$a(r) = \frac{J}{Mc} = \frac{I(r)\omega}{M(r)c}$$

In the relativistic axisymmetric case with rotation with angular velocity ω, u^{μ} has the form [11]:

$u^{\mu}=(u^0 , 0, 0, \omega u^0)$

Now u^0 is calculated from the condition

$$c^2 = g_{\mu\nu} u^{\mu} u^{\nu}$$

and the covariant velocity from

$$u_{\mu} = g_{\mu\nu} u^{\nu}$$

The resulting expression for u^0 is (A_i are the Kerr correction-factors, mass M1[r1], moment of inertia I1[r1]): (9a)

u^0 =

$$\frac{1}{\sqrt{\mathrm{A0}(\mathrm{r1},\mathrm{th})\sqrt{\left(\frac{\mathrm{r1}\,\mathrm{M1}(\mathrm{r1})^3}{\omega^2\,\mathrm{I1}(\mathrm{r1})^2\cos^2(\mathrm{th})+\mathrm{r1}^2\,\mathrm{M1}(\mathrm{r1})^2}-1\right)^2}-\frac{\mathrm{r1}\,\omega^4\,\mathrm{I1}(\mathrm{r1})^2\,\mathrm{M1}(\mathrm{r1})\cos^4(\mathrm{th})\,\mathrm{A3}(\mathrm{r1},\mathrm{th})}{\omega^2\,\mathrm{I1}(\mathrm{r1})^2\cos^2(\mathrm{th})+\mathrm{r1}^2\,\mathrm{M1}(\mathrm{r1})^2}-\frac{\omega^4\,\mathrm{I1}(\mathrm{r1})^2\sin^2(\mathrm{th})\,\mathrm{A3}(\mathrm{r1},\mathrm{th})}{\mathrm{M1}(\mathrm{r1})^2}-\mathrm{r1}^2\,\omega^2\sin^2(\mathrm{th})\,\mathrm{A3}(\mathrm{r1},\mathrm{th})+\frac{2\,\omega^2\sqrt{\mathrm{I1}(\mathrm{r1})^2}\,\mathrm{M1}(\mathrm{r1})\sqrt{\mathrm{M1}(\mathrm{r1})^2}\sin^2(\mathrm{th})\,\mathrm{A4}(\mathrm{r1},\mathrm{th})}{\omega^2\,\mathrm{I1}(\mathrm{r1})^2\cos^2(\mathrm{th})+\mathrm{r1}^2\,\mathrm{M1}(\mathrm{r1})^2}}}$$

The state equation for the pressure *P* for the nucleon gas has the form

$$P = c1\,\rho^{\gamma}$$

or in the dimensionless form with a critical density ρ_c and dimensionless pressure P_1 and density ρ_1

$$P_1 := \frac{P}{c^2 \rho_c} = k1\left(\frac{\rho}{\rho_c}\right)^{\gamma} = k1\left(\rho_1\right)^{\gamma} \tag{10}$$

For the horizon, with rotation there is the inner and the outer horizon ($M=M_0$)

$$r_- = \frac{M_0}{2} - \sqrt{\left(\frac{M_0}{2}\right)^2 - a^2}$$

$$r_+ = \frac{M_0}{2} + \sqrt{\left(\frac{M_0}{2}\right)^2 - a^2}$$

3. The equations for the extended Kerr space-time.

The solution process starts with the metric tensor $g_{\mu\nu}$ in original Eddington-Finkelstein-coordinates, with 6 non-zero components, corresponding correction-factor functions A0,…,A5, and additive correction functions B0,…B3 for the zero components.

(11)

$$g_{\mu\nu} = \begin{pmatrix} \left(1-\frac{r r_s}{\rho_{12}}\right)A0 & -A3 & B1 & -\frac{r r_s a \sin^2\theta}{\rho_{12}}A4 \\ & B0 & B2 & -a\sin^2\theta\, A5 \\ & & -\rho_{12}\,A1 & B3 \\ & & & -A2\sin^2\theta\left(r^2+a^2+\frac{r r_s a^2 \sin^2\theta}{\rho_{12}}\right) \end{pmatrix}$$

and alternatively in Boyer-Lindquist-coordinates, corresponding correction-factor functions A0,…,A4, and additive correction functions B0,…B4 for the zero components

(12)

$$g_{\mu\nu} = \begin{pmatrix} \left(1-\frac{rr_s}{\rho_{12}}\right)A0 & B0 & B1 & \frac{rr_s a\sin^2\theta}{\rho_{12}}A4 \\ & -A1\rho_{12}/\Lambda_{12} & B2 & B3 \\ & & -\rho_{12}\,A2 & B4 \\ & & & -A3\sin^2\theta(r^2+a^2+\frac{rr_s a^2\sin^2\theta}{\rho_{12}}) \end{pmatrix}$$

The equations are the 10 Einstein equations eqR00,eqR11,eqR22,eqR33,eqR12,eqR23,eqR31,eqR01,eqR02,eqR03 in the (dimensionless) variables relative radius $r_1 = \frac{r}{r_{ss}}$ and complementary azimuth angle $\theta_1 = \frac{\pi}{2} - \theta$ with energy tensor $T_{\mu\nu}$ from (8) and the state equation $P_1 = k1\left(\rho_1\right)^\gamma$ for the relative pressure P_1 and the relative density ρ_1 . We are using the so called "sun units" $r_{ss} = r_s(sun)$, $M_s = M(sun)$, $\rho_s = \frac{M_s}{4\pi\, r_{ss}^{\;3}}$, $P_s = \rho_s c^2$ for radius r, mass M, density ρ, and pressure P, respectively.

In "sun units" the original angle differential $d\Omega = 4\pi\sin\theta\, r^2 dr d\theta$ is transformed into $d\Omega = 3\cos\theta\, r^2 dr d\theta$, as for θ=0…π/2 ,r=0..1: $\int d\Omega = 1$.

Also, all equations and variables are symmetric (even) in θ : Ai(-θ)=Ai(θ) .

From now on we skip the index of the dimensionless variables and use the original notation, e.g. *r* instead of r_1 .

Furthermore, we adopt the Boyer-Lindquist coordinates and the metric tensor (12).

In sun units, the Boyer-Lindquist metric tensor becomes: (12a)

$$g_{\mu\nu} = \begin{pmatrix} \left(1-\frac{rM_0}{\rho_{12}}\right)A0 & B0 & B1 & \frac{rM_0 a\cos^2\theta}{\rho_{12}}A4 \\ & -A1\rho_{12}/\Lambda_{12} & B2 & B3 \\ & & -\rho_{12}\,A2 & B4 \\ & & & -A3\cos^2\theta(r^2+a^2+\frac{rM_0 a^2\cos^2\theta}{\rho_{12}}) \end{pmatrix}$$

$\rho_{12} = (r^2 + a^2\sin^2\theta)$

$\Lambda_{12} = r^2 - rM_0 + a^2$

where M_0 is the mass in sun units.

The 10 Einstein equations have a distinctive structure:

there are 6 primary variables A0, A2, A3, A4,B1,B4 with highest derivative ∂_{rr}

and 4 secondary variables A1,B2,B0,B3 with highest derivative ∂_r . Primary variables have boundary conditions for the variable and its r-derivative, secondary variables only for the variable itself.

This structure is dual in θ: again there are 6 θ-primary and 4 θ-secondary variables.

6 of Einstein equations contain only one 2-derivative ∂_{rr} of a primary variable:
eqR03($\partial_{rr}A4$) , eqR22($\partial_{rr}A2$) , eqR00($\partial_{rr}A0$) , eqR33($\partial_{rr}A3$) , eqR02($\partial_{rr}B1$) , eqR23($\partial_{rr}B4$)
3 contain only 2-derivatives of a secondary variable:
eqR12, eqR01, eqR31
eqR11 contains all derivatives ∂_{rr} of a primary variable.
If we make the ansatz Bi=0, several of the eqRij become identically 0, and we get the 6 equations eqR00, eqR11, eqR22, eqR33, eqR03, eqR12 for the 6 variables Ai and ρ, with the highest derivatives resp. $\partial_{rr}A0$, $\partial_{\theta\theta}A1$, $\partial_{rr}A2$, $\partial_{rr}A3$, $\partial_{rr}A4$, ($\partial_{rr}A2$, $\partial_{\theta\theta}A1$).
Thus, we are left with the 6 differential equations degree 2 in *r*, θ
non-linear (quartic) in variables Ai and their 1-derivatives and linear in ρ , ρ^{γ} .
In total, we have 6 algebro-differential eqs for 6 variables Ai and ρ (ρ enters only algebraically).
We can add 2 dependent equations eqR41== $D_{\mu}T^{\mu 1}=0$ and eqR42== $D_{\mu}T^{\mu 2}=0$, from the covariant continuity equations $D_{\mu}T^{\mu\nu}=0$, where $D_{\lambda}T^{\mu\nu}=\partial_{\lambda}T^{\mu\nu}+\Gamma^{\mu}_{\lambda\kappa}T^{\kappa\nu}+\Gamma^{\nu}_{\lambda\kappa}T^{\mu\kappa}$ is the gravitational covariant derivative.
In eqR41 ρ enters with $\partial_{r}\rho$, in eqR42 ρ enters with $\partial_{\theta}\rho$.
So, alternatively, we have the diff. equations eqR00, eqR11, eqR22, eqR33, eqR03, eqR41 , with the highest derivatives resp. $\partial_{rr}A0$, $\partial_{\theta\theta}A1$, $\partial_{rr}A2$, $\partial_{rr}A3$, $\partial_{rr}A4$, $\partial_{r}\rho$ (diff. eq. degree 1 in r for ρ)
or
the diff. equations eqR00, eqR11, eqR22, eqR33, eqR03, eqR42 , with the highest derivatives resp. $\partial_{rr}A0$, $\partial_{\theta\theta}A1$, $\partial_{rr}A2$, $\partial_{rr}A3$, $\partial_{rr}A4$, $\partial_{\theta}\rho$ (diff. eq. degree 1 in θ for ρ) .
In the Schwarzschild spacetime ω=0 and *a*=0, we have spherical symmetry, no dependence on θ , and the TOV-equation can be derived from the non-trivial eqR00, eqR11, eqR22, eqR41 .
We impose an *r*-*θ*-analytic boundary condition for Ai, $\partial_{r}Ai$, at $r=R_1$ (R_1 is the star radius) :
Ai=1, ∂_rA0=0 , ∂_rA2=0, ∂_rA3=0, ∂_rA4=0 . For A1, there is no differential boundary condition, as ∂_rA1 is the highest r-derivative, for ρ there is no boundary condition at all, because r is algebraic in the equations, but there is an integral condition:

$$M(R_1)==\int_0^{\pi/2}\int_0^{R_1}\rho(r_1,\theta)3r_1^{\,2}\cos\theta\, dr_1\, d\theta=M_0$$: integral(ρ)=total mass = M_0 .

In order to avoid the clumsy integral condition for ρ , we can introduce the mass M as a variable:

$$M(r,\theta)=\int_0^r\rho(r_1,\theta)3r_1^{\,2}\,dr_1$$

$$M(r)=\int_0^{\pi/2}M(r,\theta)d\theta=\int_0^{\pi/2}\int_0^r\rho(r_1,\theta)3r_1^{\,2}\,dr_1\cos\theta d\theta$$ is the mass of the sphere(r) and

$$\partial_r M(r,\theta)=\rho(r,\theta)\,3r^2$$

For *M(r,θ)* we impose the boundary condition at r= R_1 :
$M(R_1,\theta)=M_0$, $\partial_r M(R_1,\theta)=0$ (i.e. density ρ is zero at boundary, and total mass M_0).
So, if we take the diff. equations eqR00, eqR11, eqR22, eqR33, eqR03, eqR41 and replace *ρ(r,θ)* by $\partial_r M(r,\theta)$, we have 6 diff.equations in r,θ of degree 2, for the variables $Ai(r,\theta)$

(metric correction factors=mcf) and the mass $M(r,\theta)$, with the highest derivative $\partial_{rr}M(r,\theta)$ in M.
According to the Cauchy-Kovalevskaya theorem there exists then a unique solution in a region $R_1 > r > r_i$ within the boundary. Inside the region $r_i > r > 0$ we can enforce the vacuum Kerr-spacetime with the trivial solution $Ai=1, \rho = 0$, i.e. there is no matter there, r_i the inner radius.
The Cauchy-Kovalevskaya theorem guarantees the existence of a mathematical solution outside the horizon, but for a physical solution we must have r>=0 (meaning $\partial_r M(r,\theta)>0$) and $M(r,\theta)<=0$ for $r<= r_i$: the mass must become non-positive at the inner radius.
Therefore, for certain $\{M_0,R_1\}$ values there will be no physical solution, even for the TOV equation.

4. The solving process for the extended Kerr space-time.

In addition to the fundamental dual parameters $\{r_i , \rho_i \}$ corresponding to $\{ R_1 , M_0 \}$ in the rotation-free TOV-case, in the Kerr-case there is the new fundamental parameter Δr_i (inner ellipticity for inner boundary condition) , resp. ΔR_1 (outer ellipticity for outer boundary condition) , and the angular velocity ω. The outer radii are $R_{x1} =R_1-\Delta R_1$ and $R_{y1} =R_1$, the latter equality arising from the fact that centrifugal distortion acts only in the x-direction (the y-axis being the rotation axis). The inner radii are correspondingly $r_{xi} = r_i- \Delta r_i$, $r_{yi} = r_i$.
The **r-θ-slicing** algorithm with an Euler-step obeys the iterative procedure with slice step size h_1 in r , and step size h_2 in θ, starting with the r-boundary at $r= R_1$ (slice n=0).
The transition from slice n to $n+1$ proceeds as follows.
At slice n all variables and 1-derivatives are known from the previous step, 2-derivatives $\partial_{rr}Ai$, $\partial_{rr}Bi$ and ρ are calculated from the 6 equations.
At slice $n+1$ the variables and 1-derivatives are calculated by Euler-formula (or Runge-Kutta)

$$Ai_{n+1} = Ai_n + h_1 \partial_r Ai_n$$

$$\partial_r Ai_{n+1} = \partial_r Ai_n + h_1 \partial_{rr} Ai_n$$

The 2-derivatives $\partial_{rr}Ai$, $\partial_{rr}Bi$ and ρ are again calculated from the 6 significant equations with variables and 1-derivatives inserted from above.
The **θ-slicing r-backward** algorithm with an Euler-step obeys the iterative procedure with slice step size h_1 in θ as above for r, starting with θ=0, and solves an ordinary differential equation in r in each θ-step . The boundary condition for the r-odeq is set at $r=R_1(\theta)$ (the outer ellipse radius) with $Ai=1, M=M_0\,My0(\theta)$, $\partial_r Ai = 0$, $\partial_r M = 3(R_1)^2 \rho_{bc}$,
where ρ_{bc} is the outer boundary value for the density, ρ_{bc} =0 for the (non-interacting) neutron-gas in a shell-star and ρ_{bc} >0 , $\rho_{bc} = \rho_{equilibrium}$ for the (interacting) neutron fluid in a neutron star. $My0(\theta)$ is the mass-form-factor with the condition $\int_0^{\pi/2} My0(\theta) Cos(\theta) d\theta = 1$, i.e. the overall mass at the outer boundary is M_0 . The inner radius $r_i(\theta)$ is reached, when $My0(\theta)=0$.
The alternative (dual) **θ-slicing r-forward** algorithm starts with the boundary condition at $r= r_i(\theta)$
$Ai=1, M=0,$ $\partial_r Ai = 0$, $\partial_r M = 3(r_i)^2 \rho_{bc}$,
where $\rho_{bc} =\rho_i$ is the inner boundary value for the density, ρ_i is approximately the inner (maximum) density $\rho(r_i)$ from the corresponding TOV-equation, the value must be adapted,

so that the resulting total mass is M_0. For the compact neutron star the inner radius $r_i(\theta)$ is zero.
In the θ-slicing r-backward algorithm one starts with the outer boundary being the ellipsoid $r=R(\theta ,\Delta R_1)$, where ΔR_1 is the outer ellipticity of the star . In the θ-slicing r-forward algorithm one starts with the inner boundary being the ellipsoid $r= r_i(\theta ,\Delta r_i)$, where Δr_i is the inner ellipticity of the star .
At the inner boundary the tangential pressure is uniform, so the density is also uniform and equal to the maximum density, $\rho(\theta) =\rho_i$.
In the actual calculation we were using the θ-slicing r-backward algorithm , because here the boundary condition $M=M_0$ is achieved automatically, when one starts with $My_0(\theta)=M_0$.

The odeqs in r consist of the 6 significant Einstein equations *eqR00, eqR11, eqR22, eqR33, eqR03, eqR41* for the six variables $A0(r,\theta)$, $A1(r,\theta)$, $A2(r,\theta)$, $A3(r,\theta)$, $A4(r,\theta)$, $M(r,\theta)$ with $\theta=\theta_i$ and θ-derivatives calculated by Euler-step from the preceding q-slice. For $i=0$ i.e. $\theta=0$ the θ-derivatives are taken from start values for all variables, which normally represent the corresponding TOV-solution (here only $A0(r)$, $A1(r)$, $M(r)$ are non-trivial and do not depend on θ). The odeqs are highly non-linear algebraic differential equations and hard to solve numerically with classical methods for linear odeqs extended by an algebraic equation solver. In the case of an nonlinear odeq-system one uses an Euler or Runge-Kutta method and calculates in each step the highest derivatives with a numerical algebraic equation solver. As an alternative one can use minimization of the least-squares-error in the highest derivatives instead of a numerical algebraic equation solver. Minimization has also the advantage that one can minimize the complete set of Einstein equations plus the 2 additional continuity equations *eqR41, eqR42* in the error goal function instead of the 6 significant equations, which improves the stability of the solution (e.g. in case of degeneracy).
The **numerical error** of the algorithm is calculated from $\Sigma_i\{(eq_i)^2 ,i=1...n\}$ i.e. the Euclidean norm of the equation values (the right side of the Einstein equations being 0) from 0. The error is calculated over the lattice $\{r_i, \theta_j\}$ as median, mean or maximum. In the internal loop of the algorithm over r_i at fixed θ_j , the solution of the algebraic discretised Einstein-equation is achieved by square-root error minimization, so it is essential to avoid singularities, e.g. at the horizon and the pseudo-singularity at $\theta=0$. This is achieved by selecting appropriate analytic *convergence factors* for the (left side of) the Einstein equations . As the equations are to be zeroed for the solution, the convergence factors do not change the solution of course, but they cancel the numerical singularities, which could otherwise jeopardize the numerical convergence of the algorithm.
The **actual calculation** was carried out in Mathematica using its symbolic and numerical procedures. In the first stage, the Einstein equations were derived from the ansatz for $g_{\mu\nu}$ from section 2 and simplified automatically. The arising complexity of the equations is such, that it is practically impossible to handle them manually: the Mathematica function *LeafCount*, which returns the number of terms in the equation, gives the complexity of
LeafCount[eqR00]=17408 , LeafCount[eqR11]=27528 , LeafCount[eqR22]=134929
for the first 3 equations. To verify the equations, the TOV equation was derived by symbolic manipulation for $\omega=0$ $a=0$ from *eqR00, eqR11, eqR22, eqR41* .
The power of Mathematica is sufficient to solve the TOV equation with the single procedure *NDSolve* . For the full Einstein equations it fails even for the ordinary differential equations (odeq) in r arising for fixed θ . It took us a long time to find an algorithm, which could handle the complexity of the equations and solve them in an acceptable time (3-5 minutes

on a 16x8 lattice) on a PC-desktop and converge in the required region with an acceptable error of around *0.01*.
For the second numerical stage we tried several slicing algorithms, and the best alternative proved to be the *θ-slicing r-forward* algorithm implemented by hand in Mathematica. The solution of the resulting odeq in each r-step was calculated using *NDSolve*.
Also, for every star model and parameter set, the TOV solution with *ω=0 a=0* was calculated first with the algorithm and compared with the exact TOV solution.

5. The TOV equation as the limit ω->0 for the extended Kerr space-time.

In the Schwarzschild spacetime ω=0 and *a*=0, we have spherical symmetry, no dependence on θ , then the TOV-equation can be derived from the remaining non-trivial Einstein equations eqR00, eqR11, eqR22, eqR41 .

The TOV-equation is in the standard form:

$$P'(r) = -(\frac{GM(r)\rho(r)}{r^2})(1+\frac{P(r)}{\rho(r)c^2})(1+\frac{4\pi r^3 P(r)}{M(r)c^2})(1-\frac{2GM(r)}{rc^2})^{-1} \qquad (13)$$

and using r_s

$$P'(r) = -(\frac{c^2 r_s \rho(r)}{2r^2})(1+\frac{P(r)}{\rho(r)c^2})(1+\frac{4\pi r^3 P(r)}{M(r)c^2})(1-\frac{r_s M(r)}{rM_t})^{-1}$$, where

M_t is the total mass, furthermore

$4\pi r^2 \rho(r) = M(r)$, $P(r) = k_1 \rho(r)^\gamma$

In order to make the variables dimensionless, one introduces 'sun units'

$$r_{ss} = r_s(sun) = \frac{2GM_{sun}}{c^2} = 3km \,,\ \rho_s = \frac{M_{sun}}{4\pi r_{ss}^3/3} = 1.76 10^{16}\frac{g}{cm^3}, P_s = \rho_s c^2$$

where r_{ss} Schwarzschild-radius of the sun, ρ_s the corresponding Schwarzschild-density and P_s the corresponding Schwarzschild-pressure.
In 'sun units' TOV-equation transforms into

$$P_1'(r_1)r_1^3(r_1 - M_1(r_1)M_0) = -\frac{1}{2}(\frac{M_1'(r_1)M_0}{3} + P_1(r_1)r_1^2)(M_1(r_1)M_0 + 3P_1(r_1)r_1^3) \qquad (14)$$

with the normalized mass *M1(r1)* , and $M_1(R_1) = 1$,
or

$$P_1'(r_1)r_1^3(r_1 - M(r_1)) = -\frac{1}{2}(\frac{M'(r_1)}{3} + P_1(r_1)r_1^2)(M(r_1) + 3P_1(r_1)r_1^3)$$

where $M_0 = \frac{M_t}{M_{sun}}$, $M(r_1)$ is the mass within the radius *r*, *$M(r_1) = M_0 M_1(r_1)$*

in dimensionless variables $r_1, \rho(r_1) = \frac{M'(r_1)}{3r_1^2}, M, P_1 = k_1\rho(r_1)^\gamma$

and R_1 is the dimensionless radius of the star.
With the replacement $P = k1\,\rho^\gamma$ for the pressure from the equation of state and $\rho = \frac{M_0 M'}{3r^2}$ we obtain a diff. equation for M degree 2 in r and we impose the boundary condition in r=R_1:
$M(R_1)=M_0$, $M'(R_1)=0$ for non-interacting Fermi-gas

and
for an interacting Fermi-gas : $M(R_1)=M_0$, $M'(R_1)=\rho(R_1)\,3R_1^{\,2}, \rho(R_1)=\rho_e$, where ρ_e is the equilibrium density in the minimum of V_{nn} and $P1'(\rho_e)=0$ (here an equivalent boundary condition is $\rho'(R_1)=\infty$).

6. The equation of state and rotation parameters

6.1. The equation of state for an (non-interacting) nucleon gas

Here, $P = k1\,\rho^{\gamma}$ is the equation of state of the star, derived from the thermodynamic Fermi gas equation at T=0 ([2], chap. 48).

$$P=-\frac{\partial E}{\partial V}=8\pi P_0\left(\frac{x_F^{\;3}}{3}\sqrt{1+x_F^{\;2}}-f(x_F)\right) \qquad (15)$$

$P_0=\frac{mc^2}{\lambda_c^{\;3}}=\frac{m^4c^5}{h^3}$, where λ_c is the de-Broglie wavelength of the Fermi gas with particle mass m, $\lambda_c=\frac{h}{mc}$

$x_F=\frac{p_F}{mc}=\frac{\lambda_c}{2}(3\pi)^{1/3}n^{1/3}$, where x_F is the Fermi-angular-momentum, *n* the particle density

$$f(x_F)=\int_0^{x_F}dx\,x^2\sqrt{1+x^2}$$

The resulting approximate equations of state for *P* are

$$P=8\pi P_0\begin{pmatrix}\dfrac{x_F^{\;5}}{15}\\[2mm] \dfrac{x_F^{\;4}}{12}\end{pmatrix}=\begin{pmatrix}K_1\rho^{5/3} & \rho<<\rho_c\\ K_2\rho^{4/3} & \rho>>\rho_c\end{pmatrix} \qquad (16)$$

valid for the density ρ and the critical density ρ_c

$$\rho_c=\frac{m}{\lambda_c^{\;3}}\frac{8\pi}{3}$$

The full expression for *P*, including temperature *T*, is as follows ([4],chap.15).
Here, we use dimensionless variables (r_1 distance unit de-Broglie-wavelength λ_c , V_1 volume unit λ_c^3,n_1 particle density unit $1/\lambda_c^3$, E_1 energy unit $E_0=\frac{\hbar c}{\lambda_c}=\frac{mc^2}{2\pi}$, inverse thermal energy $\beta_1=\frac{E_0}{kT}$, chem. potential μ_1 in E_0)., for the gas model we use the Debye model with the state density $D_1(E_1)=\frac{1}{4\pi^{7/2}}\sqrt{E_1}$, maximum energy $\varepsilon_{F1}=\frac{3^{2/3}\pi^{1/3}}{4}n_1^{\;2/3}$, the resulting particle density is

$$n_1=\frac{N_{op}}{V_1}=\frac{2}{V_1}\int_0^{\infty}d\omega_1\frac{D_1(\omega_1)}{1+\exp(\beta_1(\omega_1-\mu_1))}=\frac{1}{2\pi^{7/2}}\int_0^{\infty}d\omega_1\frac{\sqrt{\omega_1}}{1+\exp(\beta_1(\omega_1-\mu_1))}$$

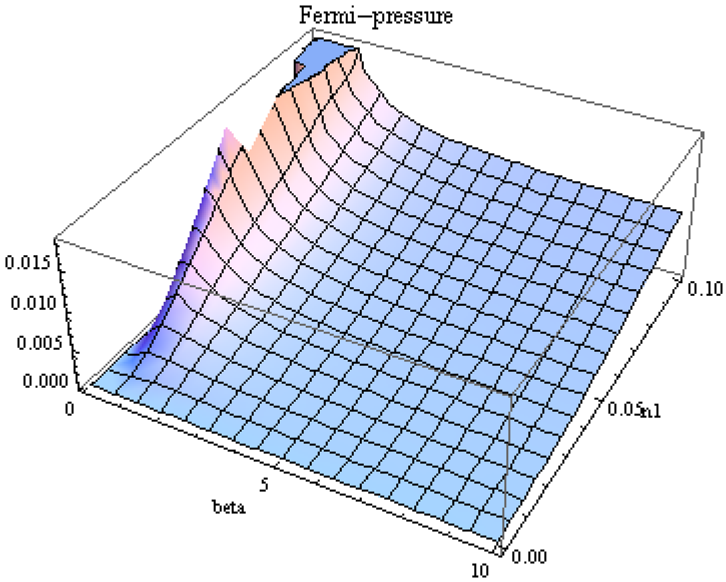
From this relation the chem. potential μ_1 can be calculated, an approximation formula is

$$\mu_1=\varepsilon_{F1}-\frac{\pi^2}{12\beta_1^{\;2}\varepsilon_{F1}}=\mu_1(n_1)$$

Finally, the resulting pressure (=energy density) $p_1(\beta_1,n_1)$:

$$p_1(\beta_1, n_1) = \frac{4\pi^{3/2}}{3\pi^2} \int_0^\infty d\omega_1 \frac{\omega_1^{3/2}}{1+\exp(\beta_1(\omega_1 - \mu_1(n_1)))} \quad (17)$$

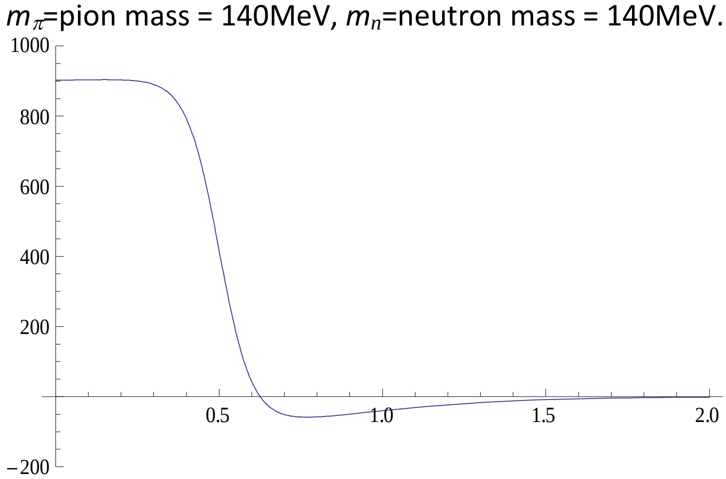
Below a 3D-diagram of $p_1(\beta_1, n_1)$ in dimensionless variables for a nucleon gas ($m=m_n$, density $\rho = \frac{E_0 n}{c^2}$ in sun units, E_0=149.4MeV) is depicted:



Here kT is in E0 units, and one sees the dependence $P = k1\,\rho^{\gamma}$ except on the left side, when kT reaches the magnitude of 1Gev (T=10^10K).

6.2. The equation of state for an (interacting) nucleon fluid

For the interacting nucleon gas we take into account the nucleon-nucleon-potential in the form of a Saxon-Woods-potential modeled on the experimental data:
`Vsw[r_,V0_,r0_,dr0_]=V0/(1+Exp[(r-r0)/dr0])`
`Vnn[r_]= Vsw[r,Va,ra,dra]+ Vsw[r,Vc,rc,drc]` where `Vnn` is the nucleon-nucleon-potential with an attractive part `Vsw[r,Va,ra,dra]` and a repulsive core `Vsw[r,Vc,rc,drc]` , the distance r between the nucleons is $r = (E_n/\rho)^{1/3}$, where E_n= $(m_n c^2/(2\pi))^{1/3} = 149.4MeV \approx m_\pi c^2$ is the nuclear energy scale m_π=pion mass = 140MeV, m_n=neutron mass = 140MeV.

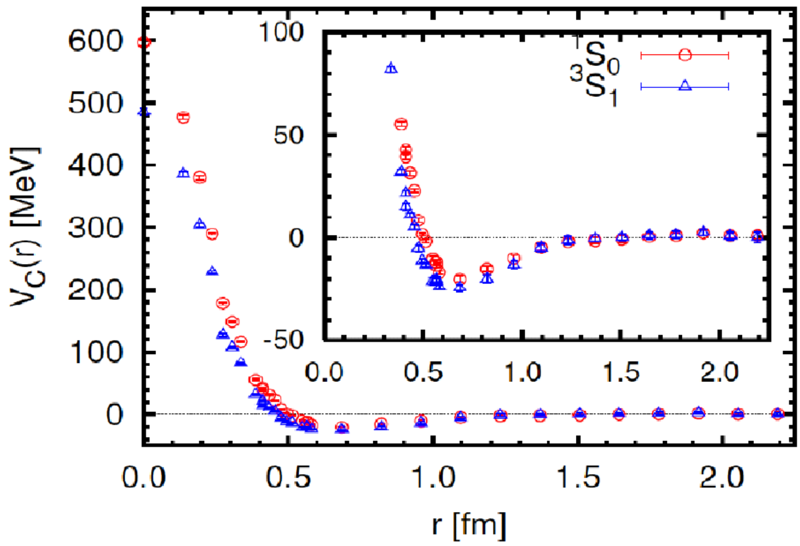


$V_{nn}(r)$ with energy(MeV), r(fm)[]

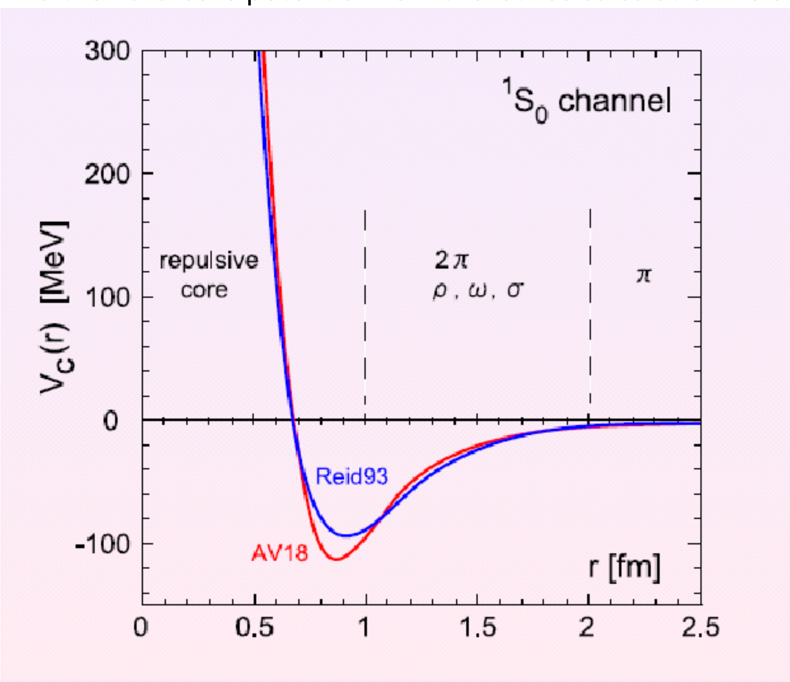
The pressure of the interacting nucleon fluid becomes then

$$P_1(r_1) = c_1 \rho_1(r_1) V_{nn}((E_n / \rho_1(r_1))^{1/3}) \tag{18}$$

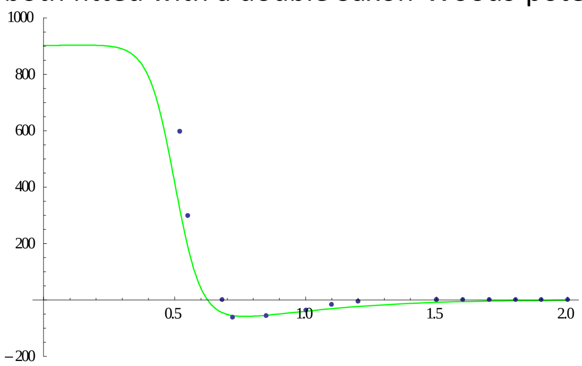
The experimental data used here are those from [7]



And the hard-core potential from the lattice calculation Reid93 from [5]



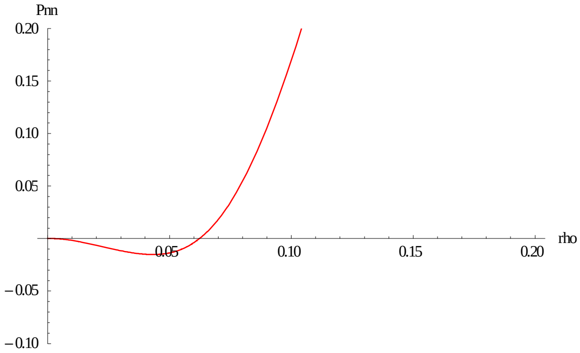
both fitted with a double Saxon-Woods-potential Vnn



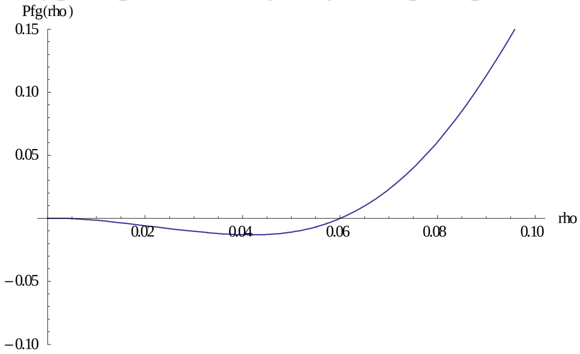
with r(fm) , V(MeV).

From the nucleon-nucleon-potential the pressure is calculated taking into account the low-density Fermi-pressure of the nucleons `K1*rho^(5/3)`

```
Pnn[rho]=Vnn[1/((rho))^(1/3)]*rho
```




```
Pfg[rho]=K1*rho^(5/3)+Pnn[rho]
```



total pressure $P_{fg}(r)$, pressure P and density r shown in sun-units.
This equation-of-state has a minimum at $\rho=\rho_c=0.0417$ and $P'(\rho)=1$ at $\rho=\rho_m=0.0544$.
As the sound velocity $v=\frac{dP(\rho)}{d\rho}$, $v>0$ and $v<1$ (i.e. subluminal), the admissible density range in the neutron-fluid model is $\rho_c<=\rho<=\rho_m$.

6.3. Maximum omega-values in Kerr-space-time

We consider here a rotation model with constant angular velocity ω. With this model the resulting 4-velocity u^μ has the form [11]:
$u^\mu=(u^0, 0, 0, \omega u^0)$

The maximum values for ω are calculated from the minimal zeros in omega of the denominator in u^0 from (9a) , minimized over $r1$ and th in their respective regions $r_i <= r_1 <= R_1$ and $0<=\theta<=\pi/2$.

The resulting value is $\omega\leq\frac{1}{2R_1\sqrt{\alpha_f}}$, where α_f is the form-factor in the moment of inertia I_1 .

$I_1=\alpha_f M R_1^{\ 2}$, α_f=2/3 for a shell , α_f=2/5 for a sphere.
With non-vanishing density the actual ω_{max} depends on $\{Ai,\rho\}$, and has to be calculated from the above expression for u^0 .
A less stringent limit for omega can be deduced from the limit for the parameter a in Kerr-spacetime (in sun-units and c=1): $a\leq\frac{r_s(M_0)}{2}=\frac{M_0}{2}$, and $a=\frac{\alpha_f M_0 R_1^{\ 2}\omega}{M_0}$ therefore

$$\omega\leq\frac{M_0}{2R_1^{\ 2}\alpha_f}$$

7. The TOV-equation: a new ansatz

Generally speaking, the parameters of the solution are :
angular momentum radius a (=alpha1, =0 for TOV), the factor in the state equation k1, the power in the state equation γ (=gam),radius R, mass M0, the relative radius uncertainty $dr02_{rel}$ (=dr02rel), the moment of inertia factor f_I (=infac, 0 for TOV), the singularity smoothing parameter ε (=epsi, see below), and the boundary factor n_{rmax} (=nrmax). Here the boundary factor enters the upper boundary of the TOV differential equation as
$r_{\max} = R(1 + n_{r\max} dr02_{rel})$.
The dimensionless TOV-equation is an differential equation in the mass M(r) of degree 2, and is highly non-linear, the dimensionless mass-density relation is $\rho = \frac{M'}{3r^2}$.
The customary way of solving the TOV equation is to impose the boundary condition at r=0 with M(0)=0, M'(0)= $3r^2\rho_0$ where ρ_0 the maximum central density .
In the new ansatz for the mass *M(r)* we impose the *outer* boundary condition at $r=R_1$:
for a pure Fermi-gas without interaction: $M(R_1)=M_0$, $M'(R_1) = \rho(R_1)\, 3R_1^{\ 2}, \rho(R_1) = 0$;
for an interacting Fermi-gas : $M(R_1)=M_0$, $M'(R_1) = \rho(R_1)\, 3R_1^{\ 2}, \rho(R_1) = \rho_e$, where ρ_e is the equilibrium density in the minimum of V_{nn} and $P1'(\rho_e)$=0 (here an equivalent boundary condition is $\rho'(R_1) = \infty$).
The star parameters mass M_0 and radius R_1 , which enter the outer boundary condition determine completely the solution. In general, there will be an *inner radius* r_i >0 with the maximum density $\rho_0 = 3r^2 M'(r_i)$ and $M(r_i)=0$. The corresponding 'dual' parameters are the inner radius r_i and the maximum density ρ_0 . One can show that for ρ_0>>ρ_c (where ρ_c is the critical density of the equation of state) there is no solution with a compact star r_i =0, i.e. there is a maximum mass M_c for the TOV equation, in case of compact neutron stars M_c = 3.04M_{sun} (see below). As we will see, there is in general a solution, if we allow r_i >0 and impose an outer boundary condition at $r=R_1$, as long as R_1 is not too close to the Schwarzschild radius r_s = M_0 of the star. In the limit $R_1 \to r_s$ there will be no positive zero of *M(r)* ,i.e. r_i <0 and the resulting (mathematical) TOV-solution will be no physical solution. But in general, speaking naively, the gravitational collapse of the star is avoided for large masses (M_0 > M_c), if it has a shell structure with the inner radius r_i and the outer radius R_1 > M_0 .
As we will see, this outer boundary condition together with allowing r_i >0 changes dramatically the resulting manifold of physical solutions.

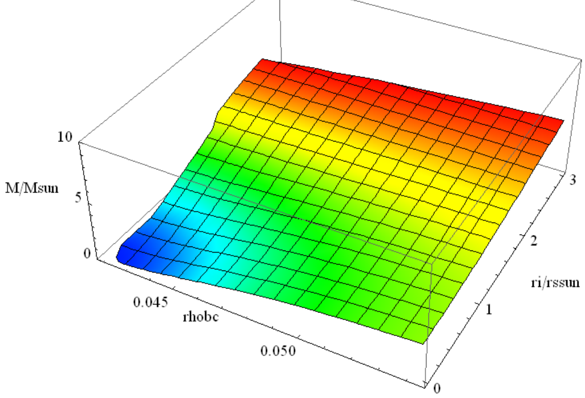
7.1. ***The TOV-equation: the parametric solution and resulting star types***

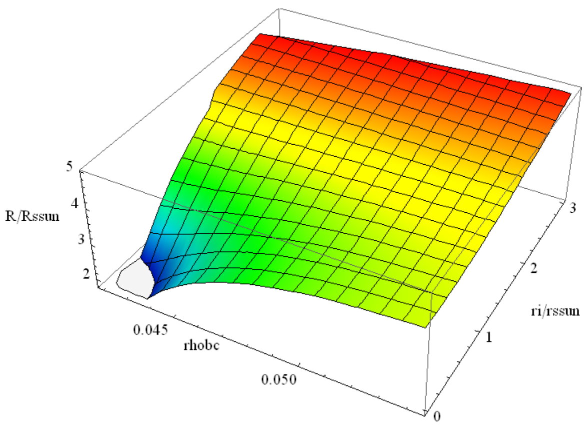
By setting-up a parametric solution of the TOV-equation one gets *a map of possible physical solutions*, i.e. possible star structures. As parameters one can use either (M_0, R_1) in the outer boundary condition at r_1= R_1 or the *dual* parameter pair (r_i, ρ_{bc}) in the inner boundary condition r_1= r_i .
The pure neutron Fermi- gas model yields for compact neutron stars a maximum mass of M_{maxc}=0.93M_{sun} , which is in disagreement with observations. Therefore, at least for compact neutron stars, a model of interacting neutron fluid must be used. In 6.2 above we have described a Saxon-Wood-potential model for the nucleon-nucleon interaction, which seems to fit the experiment and the theory in the best way. There will be a critical density (dependent on temperature of course), where a transition from interacting fluid to Fermi-gas takes place, it is plausible to set this density equal to the Saxon-Wood critical density

$\rho_c = 0.0417$.
We made calculations with the TOV-equation using these two models for neutron-based stars and we came to the conclusion that compact neutron stars with mass $M_0 <= 3.04\ M_{sun}$ consist of interacting neutron fluid and neutron shell-stars for $M_0 >= 5M_{sun}$ obey the Fermi-gas model. The underlying calculation is the Mathematica-notebook [18].
This approach yield results, which are described below.
Neutron stars consist of *interacting neutron fluid* and are compact stars with (M_0, R_1)=(0.14,1.49)...(3.04,3.95) and the maximum density $0.048 <= \rho_{bc} <= 0.0544 = \rho_{bcmax}$ in sun-units, or shell-stars with $\rho_{bc} >= \rho_{bcmax}$ and (M_0, R_1)= (3.04,3.95)…(4.91,4.92), neutron star R-M-relation follows approximately a cubic-root-law: $R \sim M^{1/3}$.
Stellar shell-stars consist of (almost) *non-interacting Fermi-gas of neutrons* and are *thin shell-stars* with $R_1 > r_s, R_1 \approx r_s, r_i \approx r_s$, i.e. the shell is close to the Schwarzschild-radius and its outer edge outside the Schwarzschild-horizon with max. density $0.0025 <= \rho_{bc} <= 0.042$, and obey an almost linear R-M-relation
(M_0, R_1)= (5.5,9.1)… (81.3,91.2), independent of ρ_{bc} for $\rho_{bc} >= 0.028$, with redshift factor around 50 for $M = 80\ M_{sun}$.
Galactic (supermassive) shell-stars are *very thin shell-stars*, which obey the *equation-of-state of a white-dwarf* (i.e. gravitation counterbalanced by Fermi-pressure of electron gas) and have an almost linear R-M-relation with redshift factor 20…100 .

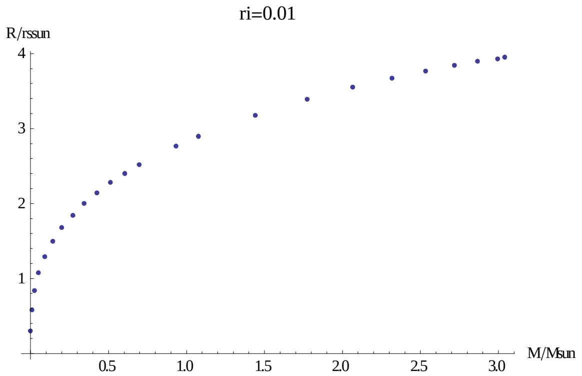
Neutron stars

The parametric solution of the TOV-equation has been carried out for the parameters (ρ_{bc}, r_i) at the boundary $r = r_i$, in the range: density $0.02 <= \rho_{bc} <= 0.15$ and inner radius $0.01 <= r_i <= 15.$, yielding physical solutions for density $0.048 <= \rho_{bc} <= 0.0544 = \rho_{bcmax}$ and inner radius $0.01 <= r_i <= 3$. The TOV-equation is solved for $M(r)$ and $\rho(r)$, and a physical solution is a mathematical solution with $M >= 0$ and $\rho >= 0$, $\rho' <= 0$ and subluminal equation-of-state within a certain interval r={ r_i , r_{02} }, which reaches a point, where $M'(r)=0$ and $\rho(r)=0$. The radius R_1 and the total mass M_0 is reached at $M'(R_1)=0$, the physical solution ends there.
The validity interval for ρ is explained by the fact, that the sound velocity $v_s(\rho) = \frac{\partial P(\rho)}{\partial \rho}$ must be positive and below 1 (subluminal in c-units).
The parametric mapping of the solutions results in the following dependence for $M_0(r_i, \rho_{bc})$, $R_1(r_i, \rho_{bc})$ (r_i, ρ_{bc}, M_0, R_1 in sun-units):



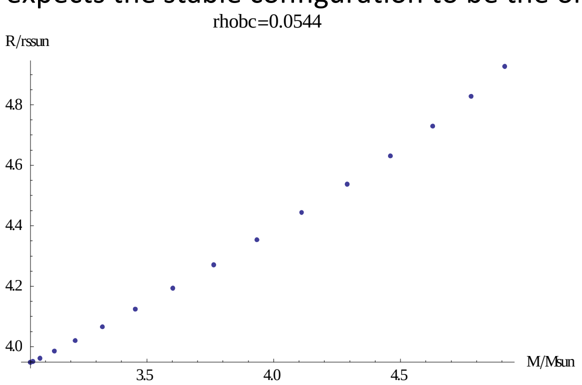


For r_i =0 the mapping describes the compact neutron stars, resulting in $R_1(M_0)$ function:

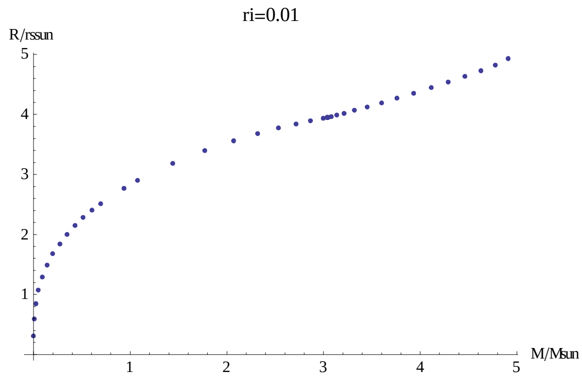


The R-M-relation follows approximately a cubic-root-law: $R \sim M^{1/3}$,
with a range of (M_0, R_1)=(0.14,1.49)...(3.04,3.95) , i.e. the resulting maximum compact mass is M_{maxc}=3.04M_{sun} .

For M0>= M_{maxc} the function $R_1(r_i=const,\rho_{bc})$ is flat or slightly decreasing with ρ_{bc} , son one expects the stable configuration to be the one with maximum $\rho_{bc} = \rho_{bcmax}$:



with a range of (M_0, R_1)= (3.04,3.95)...(4.91,4.92) .The admissible mass range ends, where the thickness of the shell above the Schwarzschild-radius becomes very small (minimum 0.01).
So in total the R-M-relation for neutron stars becomes



The maximum mass for a repulsive-hardcore-model for the equation-of-state DD2 [10] is $2.42M_{sun}$, from our mapping we have the maximum compact neutron star mass of $M_{maxc}=3.04M_{sun}$.

The actual theoretical limit for neutron star core density is $\rho_{max}=3.5\ 10^{15}$ g/cm^3=0.199 in sun-units [8,9].

The limit for ρ_{bc} reached in our mapping is only ¼ of this $\rho_{bc}=\rho_{bcmax}=0.0544$, due to the subluminal-sound-condition and the use of an (attractive) nucleon-nucleon-potential for the nucleon-fluid instead of a pure repulsive-hardcore-model.

The classical argument for the collapse of a neutron star to a black-hole for $\rho_{bc} > \rho_{max}$, dating back to Oppenheimer [2], is invalidated here by the simple introduction of shell-star models, where $r_i > 0$, and therefore there is no mass at the center, which means physically, there is only a very diluted nucleon gas there.

Stellar shell-stars

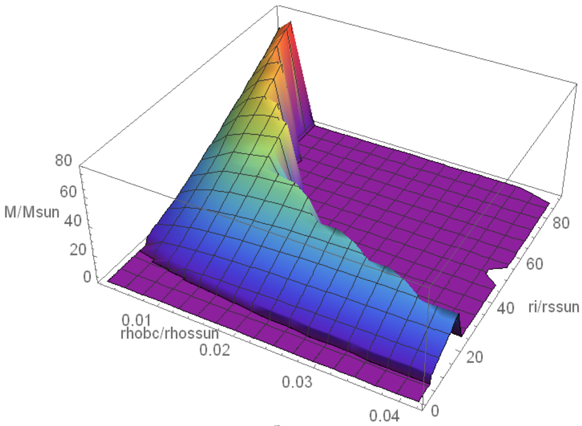
We assume that the underlying equation-of-state state for stellar shell-stars is the Fermi-gas of nucleons with the low-density limit of

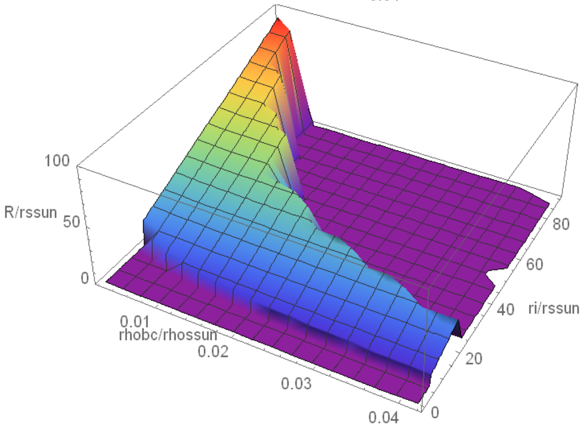
$$P(\rho) = K_1 \rho^{5/3} \ .$$

We make a further plausible assumption that the "edge" of the solution mapping are the physically stable solutions, i.e. the R-M-relation for stellar shell-stars. The edge in this case consists for fixed $\rho_{bc} < 0.0417=\rho_{oc}$ of solutions with maximum r_i (because then the average density in the shell is lowest) and for $\rho_{bc}=0.0417=\rho_{oc}$ it consists of the solutions (M_0, r_i, R_1) at the right boundary ($7 <= r_i <= 22, M_0 >= 4.91 = M_{maxn}$) , where M_{maxn} is the maximum mass for the neutron star interactive neutron fluid. We assume that at $\rho_{bc}=\rho_{oc}$ the state transition from the interactive neutron fluid to neutron Fermi gas takes place. This is almost certainly an oversimplification, still we believe that the model describes the reality correctly in principle.

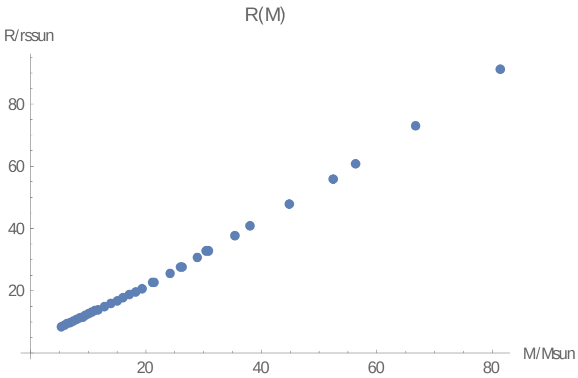
The chosen parameter range of the solution mapping is : density $0.025 <= \rho_{bc} <= 0.0417 = \rho_{oc}$ and inner radius $0.01 <= r_i <= 90$, where ρ_{oc} is equilibrium value of the nucleon-nucleon-potentials with Pnf'(ρ_{oc})=0, the transition point from the nucleon-fluid to the nucleon-gas phase .

The "edge" of the mapping yields the (M_0-, r_i ,R_1-)-range of (M_0, r_i ,R_1)= (5.35,7,.8.49)…(81.3, 91.2) , where the upper limit is in fact mathematically open, but the " thinning-out" of the solutions for small ρ_{bc} and large r_i makes it physically plausible (see the images below).

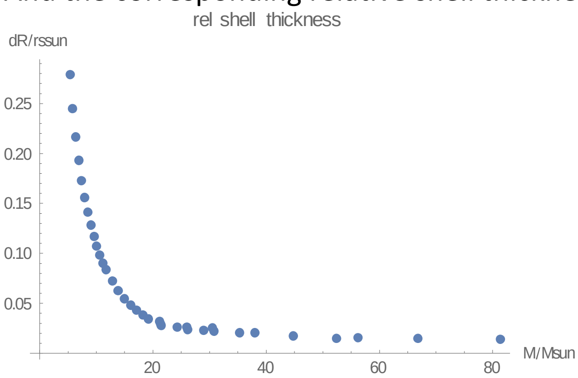




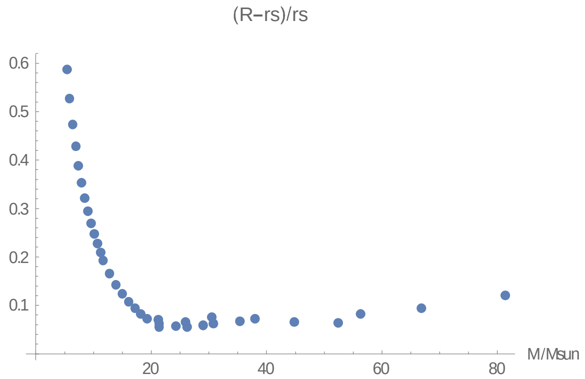
The resulting R-M-relation is practically linear and has a maximum mass value of M_{max}=81.3 M_{sun}.



And the corresponding relative shell thickness $dR_{rel}=dR/M$ is



and the relative Schwarzschild-distance $dR_{srel}=(R-M)/M$ is



The inverse of dR_{srel} gives roughly the light attenuation factor of {1.7,…50.}. Taken the attenuation factor and the small relative shell thickness of around 0.02, these stellar shell-stars have approximately the properties expected of a genuine black-hole, when measured from a distance r>>R_1 .

Entropy of a thin shell star

The celebrated Bekenstein-Hawking formula for the entropy of a black hole reads [1]:

$S = \dfrac{k_B\, A}{4{L_P}^2}$, where *A* is the surface area, k_B the Boltzmann-constant , and L_P the Planck-length.

The entropy of a (cold) shell-star with radius *R* and thickness *dR* in the limit $R=r_s$, $dR<<r_s$, with all particles in the lowest possible energy state, can be easily calculated from the Boltzmann-formula

$S = k_B \ln W$, where *W* is the number of possible micro-states.

With the elementary area πL_{min}^2 , where $L_{\min} = L_P$, *W* becomes $W = 2^{A/(\pi {L_{\min}}^2)}$ (each of the $N = A/(\pi {L_{\min}}^2)$ area elements can be occupied or empty) , so $S = \dfrac{k_B\, A}{\pi {L_P}^2} \ln 2$, which is identical to the Bekenstein-Hawking entropy with the factor *(ln2)4/π=0.882*.

Galactic (supermassive) shell-stars

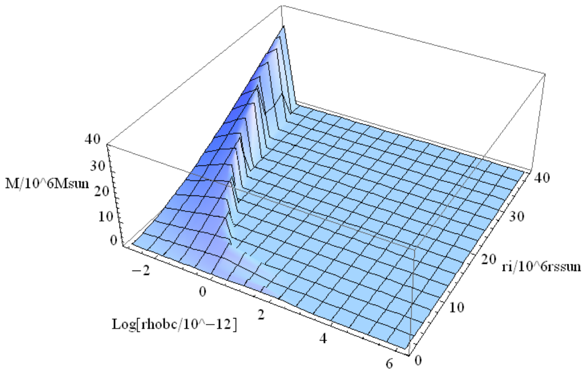
The mean density of a black-hole scales with its radius *R* like $\rho(R) = \dfrac{M}{V} = \dfrac{R}{(4/3)\pi R^3} = \dfrac{3}{4\pi R^2}$

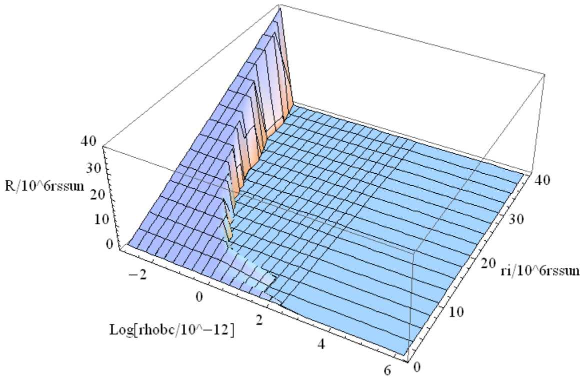
i.e. for supermassive black-hole with M=$10^6 M_{sun}$ we have $\rho \approx 10^{-12}$ in sun units (su).

In the following we use the abbreviation M M_{sun} =$10^6 M_{sun}$.

The density scale of a white-dwarf star is 10^6g/cm^3=5.7 10^{-11}su [2]. Therefore it is plausible to try a parametric mapping with the white-dwarf equation-of-state, where the underlying Fermi-pressure is that of an electron gas instead of a nucleon gas, i.e. equation-of-state $P_1(r_1) = k_1\, \rho_1(r_1)^\gamma$ for a pure Fermi gas, $\gamma=5/3$ if the density is below the critical density ρ_c .

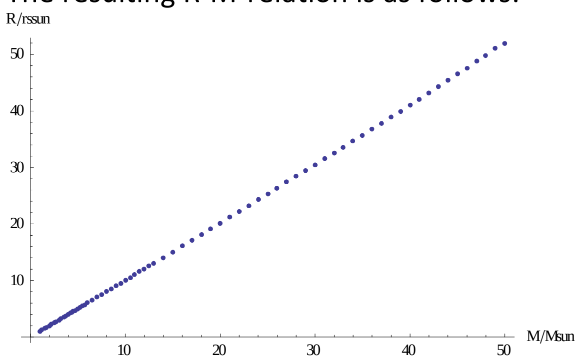
The results for $M_0(r_i,\rho_{bc})$, $R_1(r_i,\rho_{bc})$ are shown below:

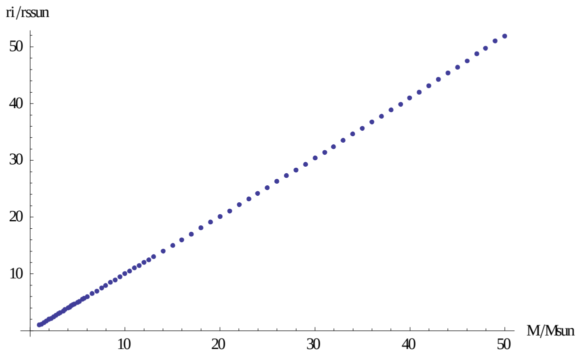


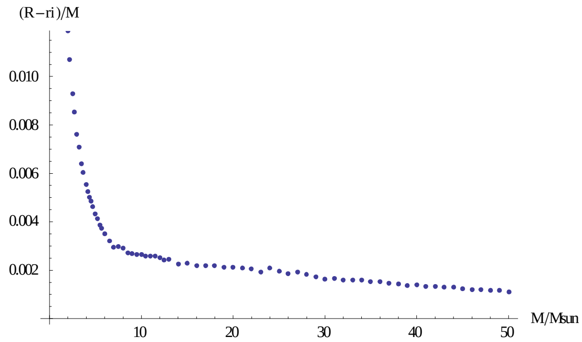


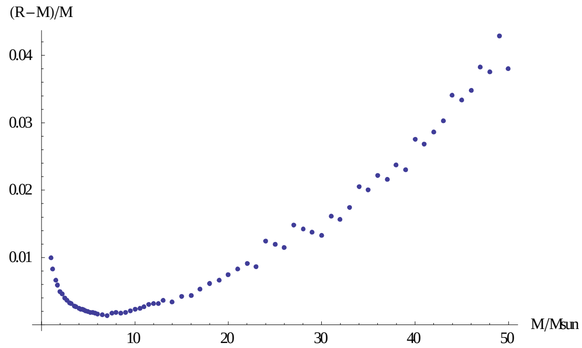
From this result one can draw several consequences: first, the actual density is around 10^{-12}, that is well below the critical density for a white-dwarf of $\rho_c=0.91\ 10^6 g/cm^3=5.17\ 10^{-11} su$: $\gamma=5/3$ in the equation-of-state is justified. Second, the viable solutions lie to the left of a "ridge" reaching up to masses around 30 MM_{sun}. Third, a stable solution for a fixed mass will have the highest possible maximum density ρ_{bc} and that will lie on the "ridge" . So one can calculate the R-M-relation following the "ridge".

The resulting R-M-relation is as follows:









The R-M-relation is almost linear, as expected, and goes up to 50MM$_{sun}$. dR$_{rel}$=(R$_1$-r$_i$)/M$_0$ is the relative thickness, and shows, that the shells are very thin indeed, with a minimum of 0.001 . The fourth diagram shows the relative Schwarzschild-distance dR$_{srel}$= (R$_1$-M$_0$)/M$_0$, which has a minimum at
{ M$_0$, dR$_{srel}$ }={7.,0.00142857}, so that its reciprocal value (approximate light attenuation factor) is around 700. So the overall result is, that the supermassive shell-stars become ever thinner shells, while the distance from the Schwarzschild-horizon is increasing.

7.2. *The TOV-equation: a case study for typical star types*

In the nearly-rotation-free case the solution of the TOV-equation was calculated for 4 models (sun units with r_s = Schwarzschild radius

$$r_{ss} = r_s(sun) = \frac{2GM_{sun}}{c^2}, \rho_s = \frac{M_{sun}}{4\pi r^3/3}, P_s = \rho_s c^2):$$

r_{ss} = 3km , ρ_s = 1.76 10^{16} g/cm^3 , M$_{sun}$ = 3 10^{30} kg ,

- average compact neutron star with mass M0= 0.932 M_{sun} , radius R1= 2.767 r_{ss}
- maximum mass neutron shell-star M0=4.91, R=4.926
- white dwarf with M0= 0.6 M_{sun} , radius R1= 3000 r_{ss}
- stellar black hole with M0= 15.69 M_{sun} , radius R1= 17.89 r_{ss} , inner radius r$_i$== 17. r_{ss}
- galactic black hole with M0= 4.367*10^6 M_{sun} , R1= 4.380*10^6 r_{ss} , r$_i$= 4.356*10^6 r_{ss}

Compact neutron star

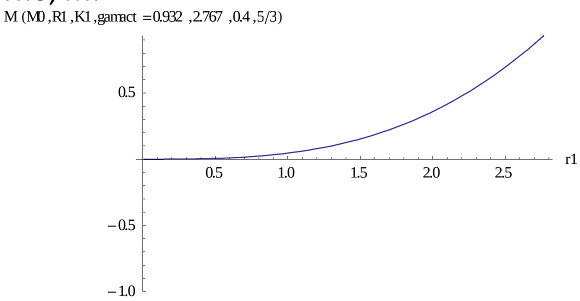
parameters= { k1=0.40,gam=5/3,M0=0.932,R1=2.76,rhobc=0.0456,ri=0.01};

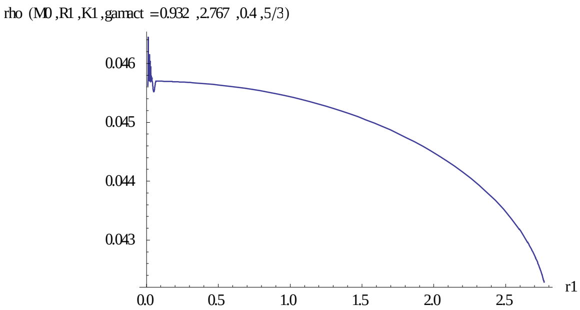
The mean density is here $\rho_{mean} = \frac{M_0}{R_1^3}$ = 0.04447 .

The critical density of the neutron Fermi gas with neutron mass m_n is $\rho_{cn} = \frac{m_n^4 c^3}{3\pi^2 \hbar^3}$ = 0.35 (see [2]) , so the low-density approximation with γ=5/3 can be used.

Results TOV:

rho, M:





As can be seen in the ρ-diagram, the derivative $\rho'(R_1) = \infty$, because there the equilibrium density ρ_c with $P'(\rho_c)=0$ in the pressure is reached.

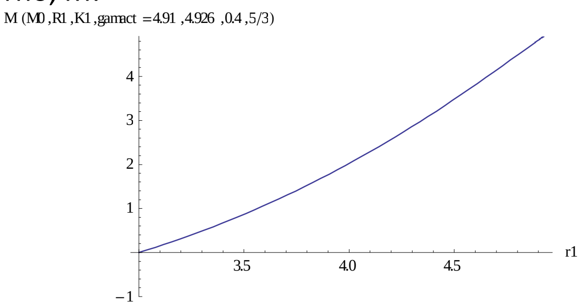
Maximum mass neutron shell-star

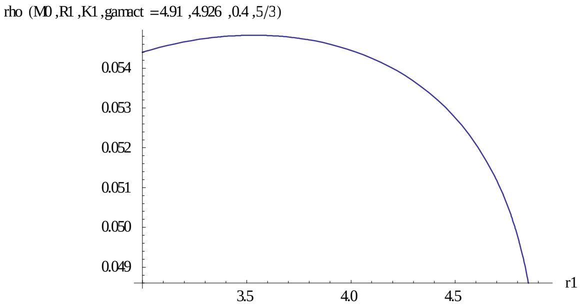
parameters= { k1=0.40,gam=5/3,M0=4.91,R1=4.926,rhobc=0.0544,ri=3.};

The mean density is here ρ_{mean} = 0.0530 .

Results TOV:

rho, M:



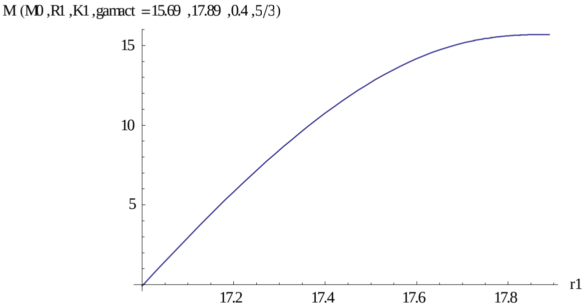


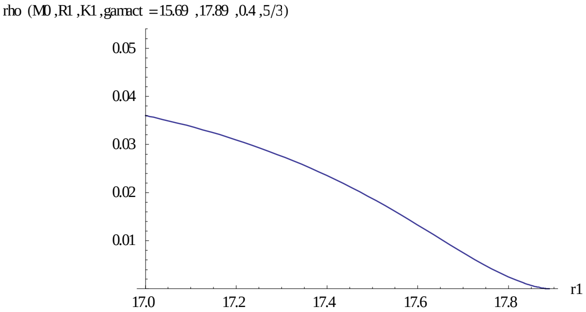
Stellar shell-star

parameters= { k1=0.40,gam=5/3,M0=15.69,R1=17.89,rhobc=0.0359,ri=17.};

The mean density is here ρ_{mean} = 0.0194 .

The resulting rho and M are:





Here the radius R_1 is reached, when $M'(r_1= R_1)=0$, i.e. $\rho(R1)=0$.

White-dwarf star

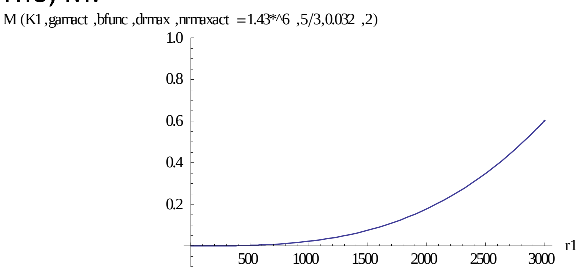
parameters= { k1=
$1.43*10^6$,gam=5/3,M0=0.6,R1=3000,rhobc=$2.02*10^{-11}$,ri=0.};

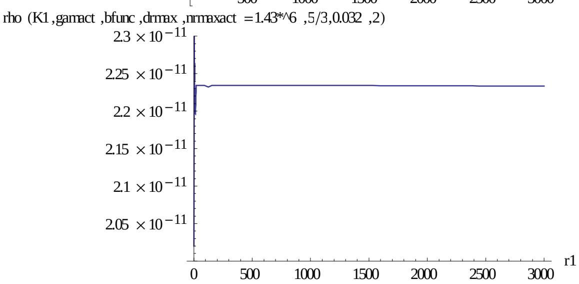
The underlying state equation is that of a small-momentum electron Fermi-gas with the critical density [2] $\rho_{cw} = \dfrac{m_e m_n{}^3 c^3}{3\pi^2\hbar^3}$ = *0.517*10^-10su .*

The mean density is here $\rho_{mean} = 2.22*10^{-11}$, the maximum deviation of ρ is $\Delta_{max}\rho = 0.21*10^{-11}$, so the density is practically constant, as expected.

The solution of the TOV-equation becomes

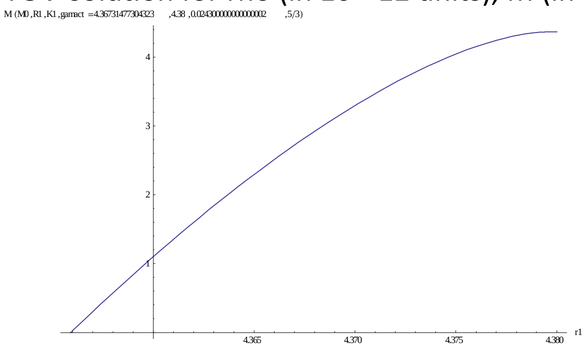
rho, M:

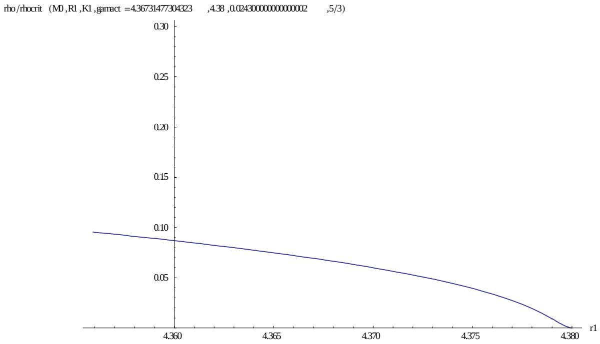




Galactic shell-star

parameters= { k11(γ=5/3)=0.0243*10^6, k12(γ=4/3)=0.067*10^4 ,M0=4.367*10^6,R1=4.380*10^6, rhobc=4.934*10^{-12},ri=4.356*10^6};
TOV equation was solved with an exterior boundary condition *r02=R_1 (M(r02)=M0, M'(r02)=0)*, which is equivalent to the interior boundary condition *r01=r_i (M(r01)=0, ρ (r01)= ρ_{bc})* , and with the full Fermi-gas equation-of-state instead of the simple power law $P(\rho) = K_1\rho^\gamma$.
The mean density is here *ρ_{mean} = 3.16*10^{-12}* .
The "naive" mean density is here $\rho_{mean} = \frac{M_0}{R_1^{\ 3}}$ =3.16*10^{-12} ,i.e. by a factor 10 lower than the mean density of white dwarf. Therefore, despite its huge mass, the galactic black hole can be described by the state equation of a small-momentum (undercritical) Fermi electron gas with the relative density $x_F = \frac{\rho}{\rho_{cn}} = 0.0612$ much smaller than that for the white dwarf.
TOV-solution for rho (in 10^-12 units), M (in 10^6 units) in r (in 10^6 units), is:





Here there is an internal "hole" with a radius r_i= 4.356*10^6, maximum ρ=4.934*10^{-12} at r_i .
The inner radius r_i lies a little below the Schwarzschild-radius r_s=M0. The relative shell thickness
$dR_{rel}=(R_1- r_i)/M_0$ =0.00551 , the relative Schwarzschild-distance $dR_{srel}=(R_1- M_0)/M_0$ =0.00290 , the light attenuation factor is roughly 1/ dR_{srel} =344 .
Furthermore, r_i is little sensitive to the temperature up to T=10^7K.
As for a stellar black hole, when R converges to $r_s=M_0$, so does the inner radius r_i , and there is no physical solution (with positive ρ and M) for a boundary within the horizon.

8. The three star models for Kerr-space-time with mass and rotation

The calculation of Kerr-space-time with mass and rotation was carried out for 3 star models:
-a typical compact neutron star with mass around 1 solar mass

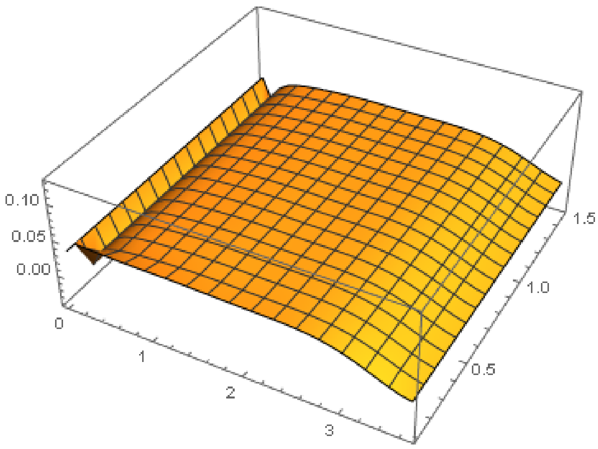
-a presumably typical stellar shell-star with a mass around 15 solar masses
-a comparatively small galactic shell-star modelled on the central black-hole in the Milky Way with a mass of around 4 million solar masses
The angular velocity ω was chosen at $0.36\omega_{max}$, i.e. about 1/3 of the maximum value.
We are using the so called "sun units" sun Schwarzschild-radius $r_{ss} = r_s(sun)$ =3km, sun mass $M_s = M(sun)$ =3. 10^{30}kg, sun Schwarzschild-density $\rho_s = \frac{M_s}{4\pi\, r_{ss}^{\;3}}$ =1.76 10^{16}g/cm^3, sun Schwarzschild-pressure $P_s = \rho_s c^2$ =1.58 10^{35}J/m^3 for radius r, mass M, density ρ, and pressure P, respectively.
The mass element here is *$M_1(r, \theta)\, dr\, d\theta$* and the ring mass *$M_1(\theta)$* is the differential mass of the θ-beam $d\theta\, M_1(\theta) = d\theta\, M_1(R_1(\theta),\theta)$, the density ρ is
$4\pi\, Cos(\theta)\rho(r,\theta)drd\theta = \partial_r \partial_\theta M_1(r,\theta)$.
As for the result values, dthrel is the maximum relative angular deviation (in θ) , and error (relative to the test function error) : wavefront error is the median (on lattice) algorithm error, the interpolation and the Fourier fit error is the error of the respective fit of the discrete solution on the lattice.
We are using here the *θ-slicing r-backward algorithm* for the shell-stars . For each of the star models a *verification step* is run first with the angular velocity ω=0, the result must be the same as in the corresponding TOV-equation. Then a parameter *case-study* is made for different outer boundary ellipticities ΔR_1 at the outer boundary condition in order to find *ΔR_1 with a minimal mean energy density*: this is the stable solution of the Kerr-Einstein equations.
The algorithm yields as the result a pointwise array of values and first and second derivatives of the variables. The variables for the 6 Einstein equations are the 5 Kerr correction-factor functions *$A0(r,\theta)$… $A4(r,\theta)$*, and *$\rho(r,\theta)$* .
We are using the *θ-slicing r-forward algorithm* for the compact neutron star. The variables for the 6 Einstein equations are the 5 Kerr correction-factor functions *$A0(r,\theta)$… $A4(r,\theta)$*, and *$M_I(r,\theta)$* .

The **value denomination** is:
a=alpha1 with $a = \frac{J}{Mc}$ the angular momentum radius (amr) of the Kerr model,
ω=omega1 is the angular velocity, R_1 = R1=r02 is the outer radius, M_0=M0 is the total mass, r1 radius variable, th angle variable,
M(r,θ)=M1(r1,th) is the mass function, A0(r,θ)… A4(r,θ) Kerr correction-factor functions,
ρ(r,θ)=rho(r1,th) is the density function,
k1 is the parameter in the approximate Fermi-gas equation-of-state $P_1(r_1) = k_1\ \rho_1(r_1)^\gamma$,
γ=gam, gam1, gam2 is the exponent ,
infac is the moment of inertia factor α_f ,
epsi is the singularity cancellation parameter with limit(epsi)=0 introduced to improve the numerical stability in singularities
ri=riact is the polar inner radius R_y
Δr_i the inner ellipticity is the difference between the polar r_{iy} and the equatorial inner radius r_{ix} , *Δr_i* = r_{iy} - r_{ix}
ΔR_1 Is the outer ellipticity , with outer radii *$R_{x1} = R_1 - \Delta R_1$* and *$R_{y1} = R_1$*
rilow is the minimal radius r1 reached in the solution

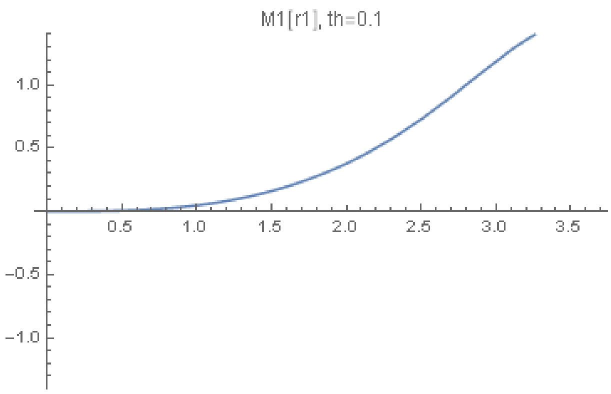
ρ_{bc}= rhobcx is the boundary condition density
dthrel is the maximum relative difference of a value dependent on θ, e.g.
dthrel(R_1)=(max($R_1(\theta)$)-min($R_1(\theta)$)-)/mean($R_1(\theta)$)
shell thickness $dR_1 = R_1(\theta)-\Delta r_i(\theta)$

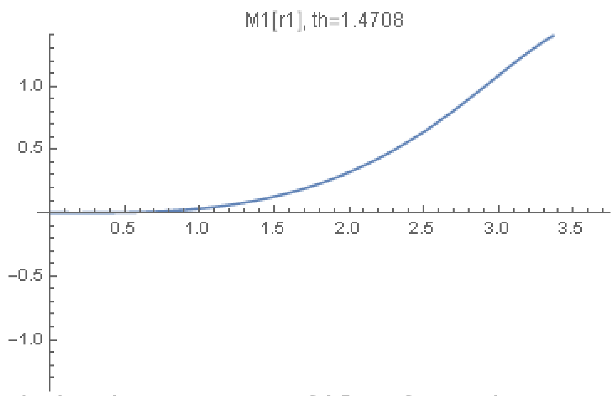
Typical compact neutron star

The underlying star model here is a compact (r_i=0) neutron star of neutron liquid (i.e. strongly interacting neutrons), mass M_0=0.932 sun-masses, radius R_1=2.76 sun-Schwarzschild-radii r_{ss} (r_{ss} =3.0km), ω=0.108688. The underlying calculation is the Mathematica-notebook [21], the results in [22].
The full parameters are:
{alpha1=0.331224,omega1=0.108688,k1=0.4,*R*1=2.7602,gam=5/3,gam1=5/3,gam2=4/3,M0=0.932,dr02rel=0.33,infac=2/5,epsi=0.1, rilow=0.001, rhobcx=0.0456}
The *r-forward* solution is first calculated with the lattice {n_x=33, n_y=17} for the rotation-free TOV-case with a TOV-solution as the initial function.
The solution for the Kerr-case starts with this corrected TOV-solution and yields the values:
outer radius $R_1(\theta)$={2.83912...2.83722},mean=2.83897, dthrel=0.00118
total mass M02eff=0.932
error: med(err)=0.0639 wavefront, =0.0491 interpolation, =0.0492 Fourier fit
mean energy density=0.0791

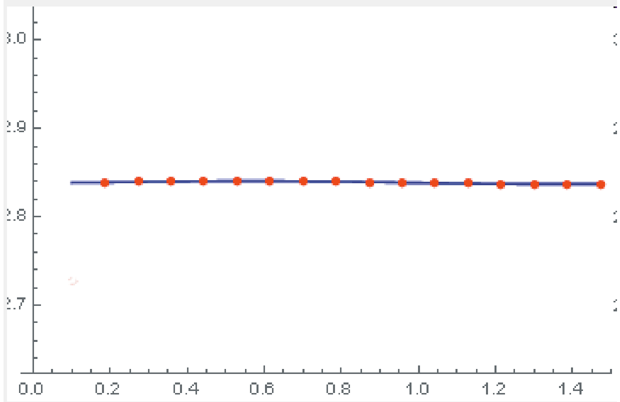


`density over x=radius r1, y=angle th`
`The density distribution is similar to the TOV-case but with a decrease in` θ`-direction.`





(ring) mass profile for th=0.1 (equatorial) and th=1.4708 (polar), the outer edge is reached when M1(r1)=M_0



effective radius r02e over angle th

The rotation results in the very small flattening in the polar direction of dthrel=0.00118. The neutron star behaves like *a fluid* because of its "viscosity", that is, its nuclear interaction and becomes "pumpkin-like".

Typical stellar shell-star

The star model here is a shell-star (r_i>Schwarzschild-radius) with mass M_0=15.69 sun-masses, radius R_1=17.89 sun-Schwarzschild-radii r_{ss} (r_{ss}=3.0km), and angular frequency ω=0.0126 . The outer Kerr-horizon is r_+=15.21 . The underlying calculation is the Mathematica-notebook [19], the results in [22].

The outer ellipticity ΔR_1 is at first a free parameter and calculated from a case-study of minimal mean energy density to ΔR_1=0.3=0.0168 R_1 .

The full parameters are

{alpha1=2.68844,omega1=0.0126,k1=0.4,R1=17.89,gam=5/3,gam1=5/3,gam2=4/3, M0=15.69,infac=2/3,epsi=0.1,rilow=15.9,rhobcx=0.036 }

The *r-backward* solution is first calculated with the lattice {n_x=17, n_y=9} for the rotation-free TOV-case as the initial function.

Then a *case study* with the parameter ellipticity ΔR_1 is carried out in order to find the minimal mean energy density, on the set of values ΔR_1 ={0,-0.3, 0.3,1.,1.6}

The case study yields a minimum at ΔR_1=0.3 (cigar-like outer boundary) , with a mean energy density=0.017004.

This *energy-minimal solution* with ΔR_1=0.3 yields the values:

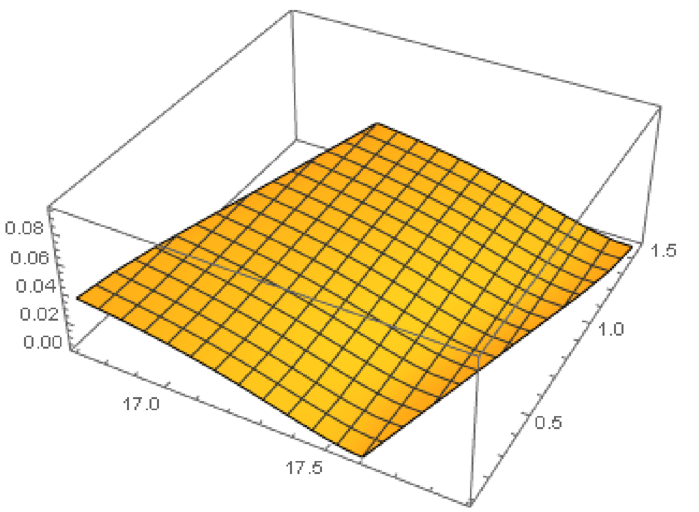
outer radius $R_1(\theta)$={17.59…17.89}, mean=17.739, dthrel=0.0164

inner radius $r_i(\theta)$={16.704…16.982}, mean=16.823, dthrel=0.0162
total mass M02eff=15.69, inner boundary max(ρ_{bc})=0.035955
shell thickness dR_1 : mean=0.916, dthrel=0.0448
mean energy density=0.0170
error: med(err)=0.00238 wavefront, =0.00573 interpolation, =0.00786 Fourier fit

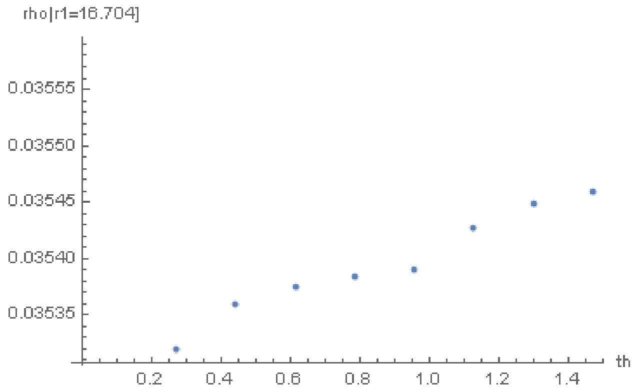
It is interesting to make a comparison with the *spherical-outer-boundary* solution with ΔR_1=0 :
inner radius $r_i(\theta)$={17.0033…17.0243}, mean=17.020, dthrel=0.00123
total mass M02eff=15.69, inner boundary max(ρ_{bc})=0.0375
shell thickness dR_1 : mean=0.870, dthrel=0.0241
mean energy density=0.017698
error: med(err)=0.00224 wavefront, =0.00545 interpolation, =0.0098 Fourier fit

The solution with the next higher ellipticity ΔR_1=1.0 has the values:
outer radius $R_1(\theta)$={16.89…17.88}, mean=17.38, dthrel=0.055
inner radius $r_i(\theta)$={16.01…16.955}, mean=16.438, dthrel=0.0556
total mass M02eff=15.69, inner boundary max(ρ_{bc})=0.0359
shell thickness dR_1 : mean=0.940, dthrel=0.0894
mean energy density=0.017249
error: med(err)=0.00309 wavefront, =0.00516 interpolation, =0.00792 Fourier fit

The two significant non-spherical features are the relative shell thickness variation dthrel(dR_1) and the relative inner ellipticity dthrel(r_i) . The first depends roughly linearly on the outer ellipticity ΔR_1 , plus the value at ΔR_1=0 (dthrel(dR_1)=0.0241) , which is results from rotation. The second, dthrel(r_i) , is almost equal to the relative outer ellipticity dthrel(r_i) , plus the small amount at ΔR_1=0 (dthrel(R_1)= 0.00123)

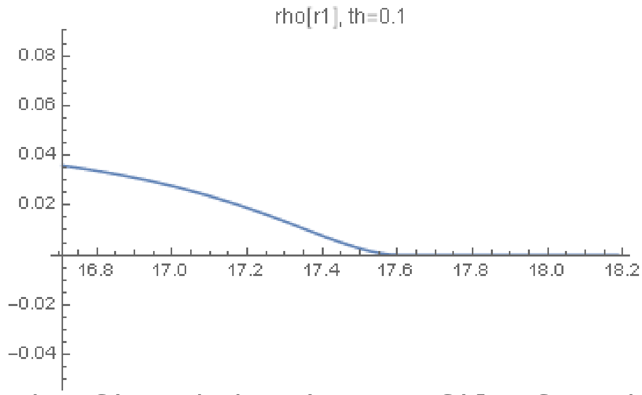


density over x=radius r1, y=angle th



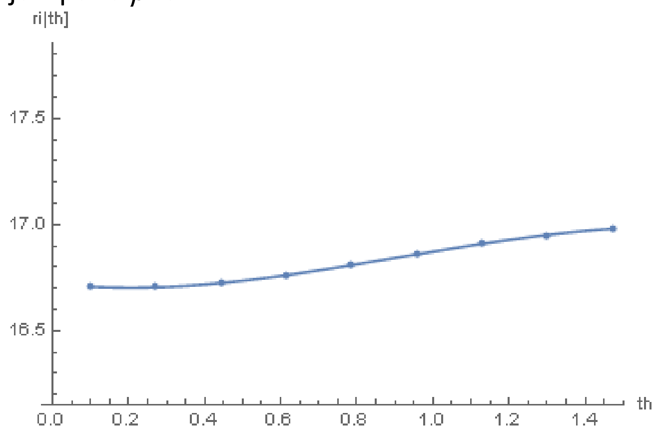
density over angle th at the inner boundary r1=16.704

The density distribution increases in θ–direction (θ=0 is equatorial, $\theta=\pi/2$ axial)

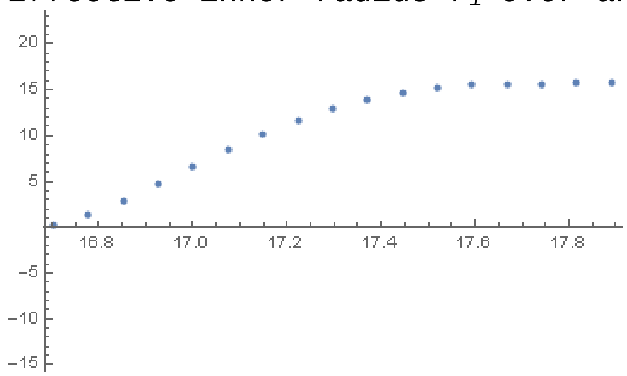


The fitted density profile for th=0.1 (equatorial)

The physical mass distribution ends at the inner boundary at r_i=16.7 , where the density jumps to $\rho=0$.



Effective inner radius r_i over angle th



Fourier-fitted total mass

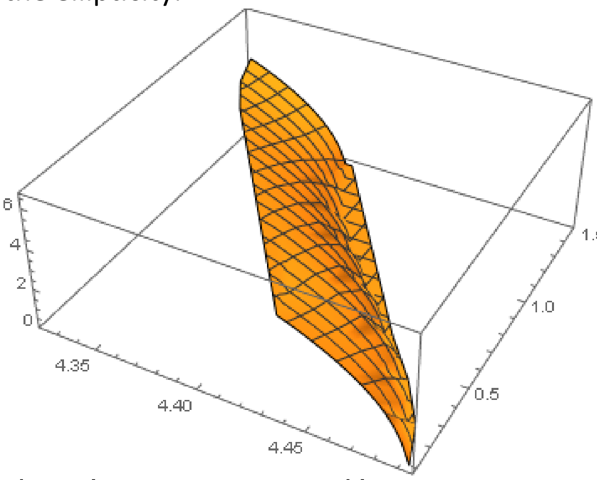
A remarkable result, distinct from the case of the neutron star, is the shape with rotation. The energy-minimal stellar shell-star behaves like a *ball of neutron gas* (negligible interaction) and decreases slightly its equatorial radius, so that, speaking naively, the increased gravitation counteracts the centrifugal force, the shell-star becomes "cigar-like", with the shell thickness approximately constant.
The stellar shell-star has all its mass concentrated within a thin shell (dR_1=0.916) which is situated outside its Schwarzschild-radius M_0=15.69, where the minimum distance from the horizon is
$min(R_1(\theta))-M_0$=1.01,
therefore the light energy loss is approximately (in Newtonian approximation) $M_0/min(R_1(\theta))$=0.9395 and the attenuation factor $1/(1- M_0/min(R_1(\theta)))$=16.53, it means that visible green light of 0.514μm is shifted to 8.50μm into middle-range infrared.

Galactic shell-star

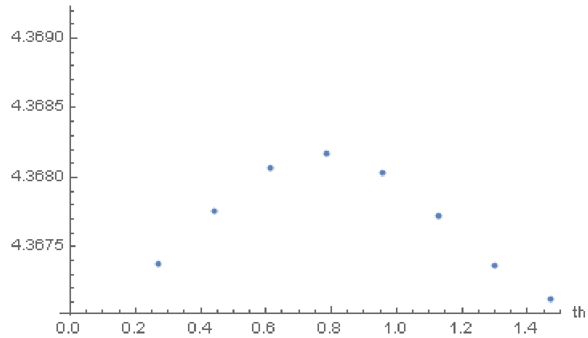
This is modelled (approximately) on the central black-hole in the Milky Way with mass M_0=4.36 mega-sun-masses (M M_s), radius R_1=4.38 mega-sun-Schwarzschild-radii (13.14 10^6km, M r_{ss}). The underlying calculation is the Mathematica-notebook [20], the results in [22].
The outer Kerr-horizon is r_+=4.26 M r_{ss} .
In order to maintain numerical performance, we are using for mass and distance 10^6 (mega) units 10^6 M_s and 10^6 r_{ss} and for density 10^{-12} (mega^{-2}) unit 10^{-12} ρ_s.
Like in the case of the stellar shell-star, the outer ellipticity ΔR_1 is at first a free parameter and calculated from a case-study of minimal mean energy density to ΔR_1=-2 dTOV, where dTOV is the shell thickness of the spherical shell-star dTOV=0.057 .
The full parameters are:
{alpha1=0.670047,omega1=0.05239,k1=0.0243,k2=0.067,R1=4.38,gam=5/3, gam1=5/3,gam2=4/3,M0=4.36731,infac=2/3,epsi=0.0024, rilow=4.3,rhobcx=4.915}
The *r-backward* solution is first calculated with the lattice {n_x=17, n_y=9} for the rotation-free TOV-case as the initial function.
Then a *case study* with the parameter ellipticity ΔR_1 is carried out in order to find the minimal mean energy density, on the set of values ΔR_1 ={0,1.,-0.3,-1.,-2.}*dTOV .
The case study yields a minimum at ΔR_1=-2 dTOV=-0.114 (pancake-like outer boundary) , with a mean energy density=1.734 .
This *energy-minimal solution* yields the values:
outer radius $R_1(\theta)$={4.494…4.38}, mean=4.43654, dthrel=0.025367
inner radius $r_i(\theta)$={4.46456…4.34109}, mean=4.40029, dthrel=0.02765
total mass M02eff=4.3673, inner boundary max(ρ_{bc})=4.96075
shell thickness dR_1 : mean=0.035256, dthrel=0.2507,
mean energy density=1.73479
error: med(err)=0.0122 wavefront, =0.000404 interpolation, =0.000437 Fourier fit

In comparison, the *spherical-outer-boundary* solution with ΔR_1=0 yields the values :
inner radius $r_i(\theta)$={4.34861…4.3423}, mean=4.343, dthrel=0.00145
total mass M02eff=4.36731, inner boundary max(ρ_{bc})=4.68324
shell thickness dR_1 : mean=0.0370, dthrel=0.1736
mean energy density=1.75914
error: med(err)=0.0101 wavefront, =0.000390 interpolation, =0.000324 Fourier fit

In contrast to the stellar shell-star, here the relative variation of the shell thickness for the spherical-outer-boundary solution is smaller by a factor of 20 as compared to the minimal solution with a high outer ellipticity, so here there is a dependence of the shell thickness on the ellipticity.

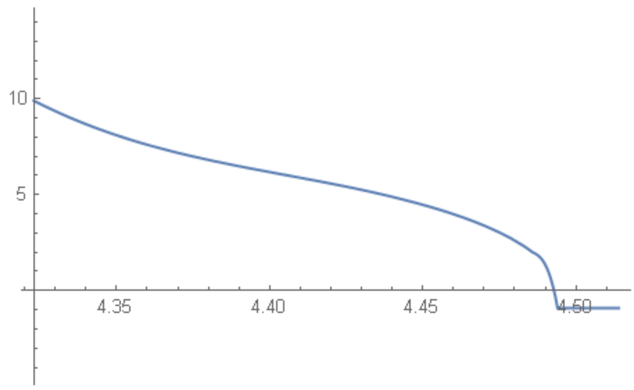


density over x=radius r1, y=angle th



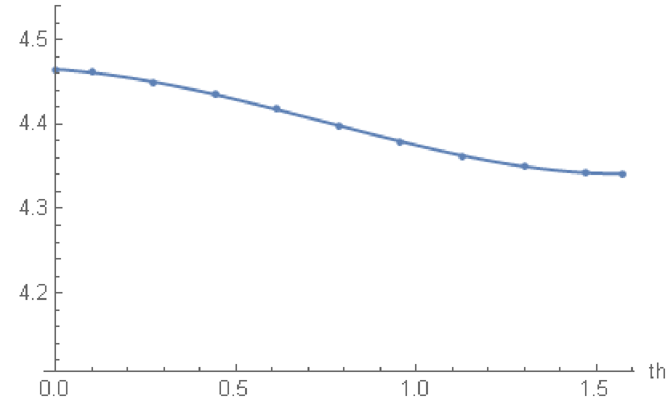
density over angle th at the inner boundary r1=4.464

The density distribution increases in ⍰–direction .

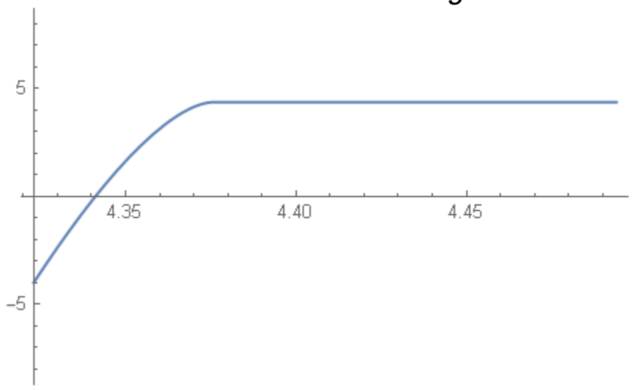


The fitted density profile for th=0.1 (equatorial)

The physical mass distribution ends at the inner boundary at r_i=4.46456, where the density jumps to $\rho=0$. The fit extrapolates it to lower r-values.



Inner radius ri over angle th



Total mass M1(r1) over radius

The maximum distance from the horizon is max(r02e)- r_+ =0.125, therefore the minimal light energy attenuation is roughly 4.262/0.125=34 , it means that visible green light of 0.514μm is shifted to 17μm into far-infrared.
The galactic shell-star has all its mass concentrated within a thin shell (dR_1= 0.0362) which has its inner radius inside and its outer radius outside its Schwarzschild-radius M_0=4.367, where the minimum distance from the horizon is $min(R_1(\theta))-M_0$=0.0127,
therefore the light energy loss is approximately (in Newtonian approximation) $M_0/min(R_1(\theta))$=0.9971 and the attenuation factor $1/(1- M_0/min(R_1(\theta)))$=345, it means that x-ray-radiation from in-falling matter from the accretion disc with an energy of 5keV and λ=0.2nm is shifted to λ=69nm, that is into hard UV-radiation.

9. Conclusions

We introduce in chap. 6 an **eos for the nucleon-fluid** in the density range $\rho_c <= \rho <= \rho_m$, where
$\rho_c=0.0417\ \rho_s$ and $\rho_m=0.0544\ \rho_s$ (sun units $r_{ss}=3km$, $\rho_s=1.76\ 10^{16}\ g/cm^3$) ,which is based on measurement data for the nucleon-nucleon-potential. This suggests, that there is a phase transition at $\rho=\rho_c$ from the (interacting) nucleon fluid to the (weakly interacting) nucleon Fermi-gas .
Based on these 2 eos's the results for the TOV-equation in chap. 7 are as follows.
Neutron stars obey the nucleon fluid eos and there are **compact neutron stars** in the range (M_0, R_1)=(0.14M_{sun},1.49 r_{ss})...(3.04 M_{sun},3.95 r_{ss}) , the R-M-relation follows approximately a cubic-root-law: $R \sim M^{1/3}$.

Neutron shell-stars exist in the range (M_0, R_1)= (3.04 M_{sun},3.95 r_{ss})…(4.91 M_{sun},4.92 r_{ss}) .
Stellar shell-stars exist in the range of (M_0, R_1)= (5.5 M_{sun},9.1 r_{ss})…(81.3 M_{sun},91.2 r_{ss}) .
The underlying equation-of-state is the Fermi-gas of nucleons with the eos $P(\rho) = K_1\rho^{5/3}$. The resulting R-M-relation is practically linear and has a maximum mass value of M_{max}=81.3M_{sun} . The light attenuation factor (redshift) is roughly {1.7…50}. Taken the redshift and the small relative shell thickness of around 0.02, these stellar shell-stars have approximately the properties expected of a genuine black-hole, when measured from a distance r>>R_1 . Furthermore, the phase space volume of a thin spherical shell is proportional to its surface A, which approximates the Bekenstein-Hawking black-hole entropy formula $S=(k_B/L_P^2)A/4$.
The **galactic (supermassive) shell-stars** have the density scale and the eos of a white-dwarf-star, i.e. of an electron Fermi-gas. The R-M-relation is almost linear and goes from 1MM_{sun} up to 50MM_{sun} (MM_{sun} =10^6M_{sun} , Mr_{ss} =10^6r_{ss}). dR_{rel}=(R_1-r_i)/M_0 is the relative thickness, and shows, that the shells are very thin indeed, with a minimum of 0.001 . The relative Schwarzschild-distance dR_{srel}= (R_1-M_0)/M_0 has a minimum at { M_0, dR_{srel} }={7. MM_{sun},0.00142857}, the redshift is around 700. So the overall result is, that the supermassive shell-stars become ever thinner shells, while the distance from the Schwarzschild-horizon is increasing.
In chap. 8 we present numerical results for rotating stars of the 3 types compact neutron star, stellar shell-star and galactic shell-star.
The angular velocity ω was chosen at ω=0.36ω_{max}, i.e. about 1/3 of the maximum.
The **compact neutron star** with M_0=0.932M_{sun}, R_{1y}=2.8372r_{ss}=8.51km , R_{1x}=2.8391r_{ss} , ω=0.1087, has the relative ellipticity of dthrel=0.00118 . The neutron star behaves like *a fluid* because of its “viscosity”, that is, its nuclear interaction, and becomes slightly “pumpkin-like”.
The **stellar shell-star** with M_0=15.74M_{sun}, R_{1mean}= 17.74 r_{ss}, ω=0.0126, has maximum density ρ_{bc}= 0.03595 ρ_s , outer radii R_{1y}=17.89r_{ss}=53.67km , R_{1x}=17.59r_{ss} , inner radii r_{iy}=16.98r_{ss} , r_{ix}=16.70 r_{ss} , inner rel. ellipticity dthrel=0.0162. The redshift is 16.53 .
The stellar shell-star behaves like a *ball of neutron gas* (negligible interaction) and decreases slightly its equatorial radius, so that, speaking naively, the increased gravitation counteracts the centrifugal force, the shell-star becomes “cigar-like”, with the shell thickness approximately constant.
The **galactic shell-star** is modelled (approximately) on the central black-hole in the Milky Way with mass M_0=4.368MM_{sun} (MM_{sun} =10^6M_{sun} , Mr_{ss} =10^6r_{ss}), radius R_1=4.38Mr_{ss} (=13.14 10^6km), ω=0.05239 .
It has maximum density ρ_{bc}= 4.961 10^{-12} ρ_s , outer radii R_{1y}=4.38Mr_{ss} , R_{1x}=4.494Mr_{ss} , inner radii r_{iy}=4.341Mr_{ss} , r_{ix}=4.464Mr_{ss} , inner rel. ellipticity dthrel=0.0276.
The redshift is roughly 345. The galactic shell-star is a shell object with a thin mass shell (ΔR=0.0352 M r_{ss}) situated close above its outer Kerr horizon r_+=4.26 Mr_{ss} . The polar radius is smaller than the equatorial radius, so the outer shape and the inner shape are both *pancake-like* .
The **overall result** is , that the introduction of numerical shell-star solutions of the TOV- and Kerr-Einstein-equations creates shell-star star models, which mimic closely the behaviour of abstract black holes and satisfy the Bekenstein_hawking entropy formula, but have finite redshifts and escape velocity v<c, no singularity , no information loss paradox, and are classical objects , which need no recourse to quantum gravity to explain their behaviour.

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