New solutions of the Tolman-Oppenheimer-Volkov equation and of Kerr space-time with matter and the corresponding star models

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Abstract
The Tolman-Oppenheimer-Volkov (TOV) equation is solved with a new ansatz: the external boundary condition with mass $M_0$ and radius $R_1$ is dual to the internal boundary condition with density $\rho_{bc}$ and inner radius $r_i$, the two boundary conditions yield the same result. The inner boundary condition is imposed with a density $\rho_{bc}$ and an inner radius $r_i$, which is zero for the compact neutron stars, but non-zero for the shell-stars: stellar quasi-black-hole and galactic quasi-black-hole. Parametric solutions are calculated for neutron stars, stellar quasi-black-holes, galactic quasi-black-holes. From the results an $M$-$R$ relation and mass limits for these star models can be extracted. A new method is found for solving the Einstein equations for Kerr space-time with matter (extended Kerr space-time), i.e. rotating matter distribution in its own gravitational field. Then numerical solutions are calculated for several astrophysical models: white dwarf, neutron star, stellar quasi-black-hole, galactic quasi-black-hole. The results suggest that quasi-black-hole star models resemble the behaviour of abstract black holes, but have finite redshifts and escape velocity $v<c$ and no singularity.

1. Introduction
In General Relativity, one of the most important applications is to calculate the mass distribution and the space-time metric for a given equation-of-state of a stellar model. Without rotation, one has spherical symmetry and in then the Tolman-Oppenheimer-Volkov (TOV) equation in radius $r$, which is derived directly from the Einstein equations (see [2]), is being used. The TOV equation consists originally of 2 coupled non-linear ordinary differential equations (odeq) of degree 1 in $r$ for mass $M(r)$ and density $\rho(r)$, where $4\pi r^2 \rho(r) = M(r)$, and can be transformed into one odeq of degree 2 for $M(r)$ by eliminating $\rho(r)$. The boundary condition is imposed normally at $r=0$ with $M=0$ and $\rho=\rho_0$, where $\rho_0$ is the maximal density.

The new ansatz presented here is the extended (inner) boundary condition at $r=r_i$ with the non-zero inner radius $r_i$, $M=0$ and $\rho=\rho_0$, i.e. the star becomes a shell-star with an (almost) void interior. With the parameters $r_i$ and $\rho_0$ this ansatz generates a 2-parametric solution manifold, where, because of energy minimization, the stable physical solution is the one with minimal $r_i$ for a given $\rho_0$ (which determines the total mass $M_0$).

The dual (outer) boundary condition is the one at $r=R$ with $M=M_0$ and $\rho=\rho_{bc}$, where $\rho_{bc}$ depends on the equation-of-state (eos): for neutron stars with interacting nucleon fluid $\rho_{bc} = \rho_c > 0$ with the equilibrium nucleon density $\rho_c$, and for the eos of non-interacting nucleon Fermi-gas (stellar quasi-black-holes) $\rho_{bc} = 0$. The 2 parameters $R$ and $M_0$ in the dual outer boundary condition correspond uniquely to the 2 parameters $r_i$ and $\rho_0$ in the inner boundary condition.
With rotation, one has an axisymmetric model in the variables \( r \) and \( \theta \) (azimuthal angle), and has to solve the Einstein equations in these 2 coordinates. In vacuum, the corresponding solution is the Kerr space-time in \( r \) and \( \theta \). With mass, a good starting point is using the extended Kerr space-time in Boyer-Lindquist coordinates with correction-factor functions \( A_0, \ldots, A_4 \) and \( B_0, \ldots, B_4 \) and the mass \( M(r, \theta) \) as variables and insert this into the Einstein equations. Setting \( B_i=0 \) some of the 10 Einstein equations become trivial and one is left with 6 partial-differential equations (pdeq) in \( r \) and \( \theta \) for the 6 variables \( A_0, \ldots, A_4 \) and \( M \).

The (outer) boundary condition here at the effective star radius \( R \) with total mass \( M_0 \) is: \( A_i=1 \), \( M=M_0 \) and \( \partial_r A_i=0 \), \( \partial_r M=0 \), as the density becomes 0 and the space-time becomes the normal Kerr space-time in vacuum.

Now, with rotation, we have a new model parameter, the angular velocity \( \omega \), to which corresponds a third parameter in the outer boundary condition: (outer) ellipticity \( \Delta R_1 \), where \( R_{3x}=R_{3y}=\Delta R_1 \) and \( R_{3x}, R_{3y} \) are the equatorial and the polar radius. As in the TOV-case, here to the 3 parameters \( R_{3y}, M_0 \) and \( \Delta R_1 \) correspond the 3 inner parameters \( r_{iy}, \rho_0 \) and \( \Delta r_i \). So here we get a 3-parametric solution manifold, and as in the spherical case, for a given total mass \( M_0 \) we have to find the stable physical solution. As before, these will be the ones with minimal \( r_{iy} \) and among them the one with minimal mean energy density: this defines the inner ellipticity \( \Delta r_i \). In all considered cases, it can be shown numerically, that such a (non-trivial) minimum exists.

The paper is organized as follows.

In 2 we present the mathematical setup, in 3 the equations for the extended Kerr space-time with rotation, in 4 the solution algorithm for it. In 5 the TOV-equation is introduced, in 6 the equation-of-state for the nucleon fluid and nucleon gas. In 7 the results for the TOV-equation are shown: the parametric solution manifold in 7.1. and the case study for typical stars in 7.2. In 8 the results for the extended Kerr space-time with rotation are presented for three typical star configurations: compact neutron star, stellar quasi-black-hole, galactic quasi-black-hole.

### 2. The Kerr space-time, Schwarzschild space-time, Einstein equations

Using the Minkowski metric \( \eta_{\mu\nu} = \text{diag}(1,-1,-1,-1) \), the Kerr space-time metric in original Kerr coordinates \((u, \theta, \phi)\) has the line element [2]

\[
d s^2 = \left(1 - \frac{r_s}{r + a^2 \cos^2 \theta}\right) \left(du + a \sin^2 \theta d\phi\right)^2 - 2 \left(du + a \sin^2 \theta d\phi\right) \left(dr + a \sin^2 \theta d\phi\right) - \left(r^2 + a^2 \cos^2 \theta\right) d\theta^2 + \sin^2 \theta d\phi^2
\]

where \( r_s = \frac{2GM}{c^2} \) is the Schwarzschild radius, and \( a = \frac{J}{Mc} \) is the angular momentum radius (amr), \( a \) has the dimension of a distance: \([a] = [r]\), and \( J \) is the angular momentum.

With this line element the Kerr metric tensor \( g_{\mu\nu} \) is as follows:

\[
g_{\mu\nu} = \begin{pmatrix}
1 + \frac{r_s}{\rho_{12}} & -1 & 0 & -\frac{rr_s a \sin^2 \theta}{\rho_{12}} \\
0 & 0 & -\rho_{12} & -a \sin^2 \theta \\
-\rho_{12} & 0 & \rho_{12} & 0 \\
0 & 0 & 0 & \eta_{33}
\end{pmatrix}
\]

with the abbreviations \( \rho_{12} = (r^2 + a^2 \cos^2 \theta) \).
and  \( g_{33} = -\sin^2 \theta (r^2 + a^2 + \frac{rr_s a^2 \sin^2 \theta}{\rho_{12}}) \).

In the limit \( a \to 0 \) the Schwarzschild space-time in advanced Eddington-Finkelstein coordinates emerges:

\[
\begin{align*}
\text{(3)} \quad ds^2 &= \left(1 - \frac{r_s}{r}\right) du^2 - 2 du\,dr - r^2 (d\theta^2 + \sin^2 \theta \,d\phi)
\end{align*}
\]

This form of the Schwarzschild line element has the advantage in comparison with the original line element

\[
\begin{align*}
\text{(3a)} \quad ds^2 &= \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r_s}{r}\right)} - r^2 (d\theta^2 + \sin^2 \theta \,d\phi)
\end{align*}
\]

that the (apparent) singularity at \( r = r_s \) is missing.

The same is valid for the original Kerr space-time: the denominator \( \rho_{12} \) has no zeros, there is no singularity in \( g_{ab} \), which makes it more well-behaved numerically. Alternatively, in Boyer-Lindquist-coordinates:

\[
\begin{align*}
\text{(4)} \quad g_{\mu\nu} &= \begin{pmatrix}
1 - \frac{rr_s}{\rho_{12}} & 0 & 0 & \frac{rr_s a \sin^2 \theta}{\rho_{12}} \\
0 & -\rho_{12} / \Lambda_{12} & 0 & 0 \\
0 & 0 & -\rho_{12} & 0 \\
-\sin^2 \theta (r^2 + a^2 + \frac{rr_s a^2 \sin^2 \theta}{\rho_{12}}) & 0 & 0 & -\sin^2 \theta (r^2 + a^2 + \frac{rr_s a^2 \sin^2 \theta}{\rho_{12}})
\end{pmatrix}
\end{align*}
\]

with the line element

\[
\begin{align*}
\text{(1a)} \quad ds^2 &= \left(1 - \frac{rr_s}{r^2 + a^2 \cos^2 \theta}\right) (dt)^2 + \left(\frac{2rr_s a \sin^2 \theta}{r^2 + a^2 \cos^2 \theta}\right) dt \,d\phi
\end{align*}
\]

\[
\begin{align*}
&- \left(\frac{r^2 + a^2 \cos^2 \theta}{r^2 - rr_s + a^2}\right) dr^2 - \\
&\left(\frac{r^2 + a^2 + \frac{rr_s a^2 \sin^2 \theta}{r^2 + a^2 \cos^2 \theta}}{r^2 + a^2 \cos^2 \theta}\right) \sin^2 \theta \,d\phi^2 - \left(\frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2 \cos^2 \theta}\right) (d\theta^2)
\end{align*}
\]

with the abbreviation \( \Lambda_{12} = r^2 - rr_s + a^2 \). Here, \( \Lambda_{12} \) has zeros at the inner/outer horizon \( r = (r_s/2) \pm \sqrt{(r_s/2)^2 - a^2 \cos^2 \theta} \), so for numerical calculations the singularity has to be removed by adding a small \( \varepsilon \): \( \Lambda_{12, \varepsilon} = \sqrt{(r_s^2 - rr_s + a^2)^2 + \varepsilon^2} \).

In the limit \( a \to 0 \) the Schwarzschild space-time in the standard form (4) emerges.

The Einstein field equations with the above Minkowski metric are:

\[
\begin{align*}
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R_0 - \Lambda g_{\mu\nu} &= -\kappa T_{\mu\nu}
\end{align*}
\]

where \( R_{\mu\nu} \) is the Ricci tensor, \( R_0 \) the Ricci curvature, \( \kappa = \frac{8 \pi G}{c^4} \), \( T_{\mu\nu} \) the energy-momentum tensor, \( \Lambda \) is the cosmological constant (in the following neglected, i.e. set 0), with the Christoffel symbols (second kind)
\[
\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda \kappa} \left( \frac{\partial g_{\kappa\mu}}{\partial \chi^\nu} + \frac{\partial g_{\kappa\nu}}{\partial \chi^\mu} - \frac{\partial g_{\mu\nu}}{\partial \chi^\kappa} \right)
\]

(6)

and the Ricci tensor
\[
R_{\mu\nu} = \frac{\partial \Gamma_{\mu\rho}^\sigma}{\partial \chi^\nu} - \frac{\partial \Gamma_{\mu\nu}^\rho}{\partial \chi^\rho} + \Gamma_{\mu\rho}^\sigma \Gamma_{\sigma\nu}^\rho - \Gamma_{\mu\nu}^\sigma \Gamma_{\sigma\rho}^\rho
\]

(7)

The crucial part of the extended Kerr solution is the expression for the energy-momentum tensor \( T_{\mu\nu} \). As usual, one uses the formula for the perfect fluid [2,(45.3)]:
\[
T_{\mu\nu} = \left( \rho + \frac{P}{c^2} \right) u_\mu u_\nu - P g_{\mu\nu}
\]

(8)

where \( P \) and \( \rho \) is the pressure and density, \( u_\mu \) is the covariant velocity 4-vector.

In the Schwarzschild case, when deriving the TOV-equation, one sets the spatial contravariant velocity components to 0: \( u^i = 0 \), in the Kerr case the tangential velocity \( u^3 = u^\phi \neq 0 \).

For the velocity one has:
\[
u^3 = r \omega = r \frac{a M c}{I}, \text{ where } I \text{ is the moment of inertia and } \omega \text{ the angular velocity,}
\]
\[
I = f_1 M R_1^2, \text{ so }
\]
\[
u^3 = \frac{a c r}{f_1 R_1^2}
\]

for a homogeneous sphere \( I = \frac{2}{5} M R^2 \) i.e. \( f_1 = \frac{2}{5} \), for a thin shell \( I = M R^2 \), for a disc
\[
I = \frac{1}{2} M R^2.
\]

If we make the obvious assumption that the star rotates as a whole, i.e. with constant angular velocity, then the moment of inertia \( I \) becomes \( r \)-dependent, like the mass \( M \):
\[
M(r) = \int_0^r \rho(r_i) 3r_i^2 \, dr_i
\]
\[
I(r) = \int_0^r \rho(r_i) 3r_i^4 \, dr_i
\]

The factor 3 in the integral instead of the usual 4\( \pi \) comes from the dimensionless calculation in “sun units” (see below).

The amr \( a \) also becomes \( r \)-dependent:
\[
a(r) = \frac{J}{M c} = \frac{I(r) \omega}{M(r) c}
\]

In the relativistic axisymmetric case with rotation with angular velocity \( \omega \) \( u^\mu \) has the form [11]:
\[
u^\mu = (u^0, 0, 0, \omega u^\phi)
\]

Now \( u^0 \) is calculated from the condition
\[
c^2 = g_{\mu\nu} u^\mu u^\nu
\]

and the covariant velocity from
\[
u_\mu = g_{\mu\nu} u^\nu
\]

The resulting expression for \( u^0 \) is (\( \epsilon \) is the singularity cancellation parameter, limit(\( \epsilon \))=0, \( A_i \) are the Kerr correction-factors, mass \( M1[r1] \), moment of inertia \( I1[r1] \)): 

(9a)
The state equation for the pressure $P$ for the nucleon gas has the form

$$P = \rho \gamma$$

or in the dimensionless form with a critical density $\rho_c$ and dimensionless pressure $P_1$ and density $\rho_1$

$$P_1 = \frac{P}{\rho_c \gamma} = k_1 \left( \frac{\rho}{\rho_c} \right)^\gamma = k_1 (\rho_1)^\gamma$$

(10)

For the horizon, with rotation there is the inner and the outer horizon ($M=M_0$)

$$r_+ = \frac{M_0}{2} \sqrt{\left( \frac{M_0}{2} \right)^2 - a^2}$$
$$r_- = \frac{M_0}{2} + \sqrt{\left( \frac{M_0}{2} \right)^2 - a^2}$$

3. The equations for the extended Kerr space-time.

The solution process starts with the metric tensor $g_{\mu\nu}$ in original Eddington-Finkelstein-coordinates, with 6 non-zero components, corresponding correction-factor functions $A_0,..,A_5$, and additive correction functions $B_0,..,B_3$ for the zero components.

(11)

$$g_{\mu\nu} = \begin{pmatrix}
    \left(1 - \frac{r_-}{\rho_{12}}\right) A_0 & - A_3 & B_1 & - \frac{rr_a \sin^2 \theta}{\rho_{12}} A_4 \\
    B_0 & B_2 & - a \sin^2 \theta A_5 & B_3 \\
    -\rho_{12} A_1 & - A_2 \sin^2 \theta (r^2 + a^2 + \frac{rr_a^2 \sin^2 \theta}{\rho_{12}}) & B_4 & B_5
\end{pmatrix}$$

and alternatively in Boyer-Lindquist-coordinates, corresponding correction-factor functions $A_0,..,A_4$, and additive correction functions $B_0,..,B_4$ for the zero components

(12)
The equations are the 10 Einstein equations eqR00, eqR11, eqR22, eqR33, eqR12, eqR23, eqR31, eqR01, eqR02, eqR03 in the (dimensionless)
variables relative radius \( r_i = \frac{r}{r_{ss}} \) and complementary azimuth angle \( \theta_i = \frac{\pi}{2} - \theta \) with energy
tensor \( T_{\mu\nu} \) from (8) and the state equation \( P_i = k_1 (\rho_i)^\gamma \) for the relative pressure \( P_1 \) and
the relative density \( \rho_1 \). We are using the so called “sun units” \( r_{ss} = r_s \,(\text{sun}) \),
\( M_s = M \,(\text{sun}) \), \( \rho_s = \frac{M}{4\pi r_{ss}^3} \), \( P_s = \rho_s c^2 \) for radius \( r \), mass \( M \), density \( \rho \), and pressure \( P \),
respectively.
In “sun units” the original angle differential \( d\Omega = 4\pi \sin \theta \, r^2 \, dr \, d\theta \) is transformed into
\( d\Omega = 3 \cos \theta \, r^2 \, dr \, d\theta \), as for \( \theta = 0 \ldots \pi/2 \), \( r = 0 \ldots 1 \):
\[ \int d\Omega = 1. \]
Also, all equations and variables are symmetric (even) in \( \theta \) : \( \text{Ai}(\theta) = \text{Ai}(\theta) \).
From now on we skip the index of the dimensionless variables and use the original notation, e.g. \( r \) instead of \( r_i \).
Furthermore, we adopt the Boyer-Lindquist coordinates and the metric tensor (12).
In sun units, the Boyer-Lindquist metric tensor becomes:
\[
g_{\mu\nu} = \begin{pmatrix}
1 - \frac{r_{M_0}}{\rho_{12}} & B0 & B1 & \frac{r r_a \sin^2 \theta}{\rho_{12}} A4 \\
- A1 \rho_{12} / \Lambda_{12} & B2 & B3 & B4 \\
- \rho_{12} A2 & & & - A3 \sin^2 \theta (r^2 + a^2 + \frac{r r_a a^2 \sin^2 \theta}{\rho_{12}}) \\
\end{pmatrix}
\]
\( \rho_{12} = (r^2 + a^2 \sin^2 \theta) \)
\( \Lambda_{12} = r^2 - r M_0 + a^2 \)
where \( M_0 \) is the mass in sun units.
The 10 Einstein equations have a distinctive structure:
there are 6 primary variables \( A0, A2, A3, A4, B1, B4 \) with highest derivative \( \partial_{rr} \)
and 4 secondary variables \( A1, B2, B0, B3 \) with highest derivative \( \partial_r \). Primary variables have
boundary conditions for the variable and its \( r \)-derivative, secondary variables only for the
variable itself.
This structure is dual in \( \theta \) : again there are 6 \( \theta \)-primary and 4 \( \theta \)-secondary variables.
6 of Einstein equations contain only one 2-derivative \( \partial_{rr} \) of a primary variable:
eqR03(\partial_\theta A4), eqR22(\partial_\theta A2), eqR00(\partial_\theta A0), eqR33(\partial_\theta A3), eqR02(\partial_\theta B1), eqR23(\partial_\theta B4)

3 contain only 2-derivatives of a secondary variable:
eqR12, eqR01, eqR31

\text{eqR11 contains all derivatives of a primary variable.}

If we make the ansatz Bi=0, several of the eqRij become identically 0, and we get the 6 equations eqR00, eqR11, eqR22, eqR33, eqR03, eqR12 for the 6 variables Ai and ρ, with the highest derivatives resp. \partial_\theta A0, \partial_\theta A1, \partial_\theta A2, \partial_\theta A3, \partial_\theta A4, (\partial_\theta A2, \partial_\theta A1).

Thus, we are left with the 6 differential equations degree 2 in \textit{r, \theta} non-linear (quartic) in variables Ai and their 1-derivatives and linear in ρ, γρ.

In total, we have 6 algebro-differential eqs for 6 variables Ai and (ρ enters only algebraically).

We can add 2 dependent equations eqR41== eqR42== 0, from the covariant continuity equations 0 = \Gamma_{\alpha\kappa}^\mu T^{\mu\nu} + \Gamma_{\nu\kappa}^\mu T^{\mu\nu} + \Gamma_{\kappa\nu}^\mu T^{\mu\nu} is the gravitational covariant derivative.

In eqR41 ρ enters with \partial_\theta ρ, in eqR42 ρ enters with \partial_\theta ρ.

So, alternatively, we have the diff. equations eqR00, eqR11, eqR22, eqR33, eqR03, eqR41 , with the highest derivatives resp. \partial_\theta A0, \partial_\theta A1, \partial_\theta A2, \partial_\theta A3, \partial_\theta A4, \partial_\theta ρ (diff. eq. degree 1 in \textit{r} for ρ)

or

the diff. equations eqR00, eqR11, eqR22, eqR33, eqR03, eqR42 , with the highest derivatives resp. \partial_\theta A0, \partial_\theta A1, \partial_\theta A2, \partial_\theta A3, \partial_\theta A4, \partial_\theta ρ (diff. eq. degree 1 in \textit{θ} for ρ).

In the Schwarzschild spacetime ω=0 and a=0, we have spherical symmetry, no dependence on \textit{θ} , and the TOV-equation can be derived from the non-trivial eqR00, eqR11, eqR22, eqR41.

We impose an \textit{r-\theta}-analytic boundary condition for Ai, \partial_\theta Ai , at \textit{r=R_1} (R_1 is the star radius) :
Ai=1, \partial_\theta A0=0, \partial_\theta A2=0, \partial_\theta A3=0, \partial_\theta A4=0 . For A1, there is no differential boundary condition, as \partial_\theta A1 is the highest \textit{r}-derivative, for ρ there is no boundary condition at all, because \textit{r} is algebraic in the equations, but there is an integral condition:

\begin{align*}
M(R_1) = \int_0^{\pi/2} \int_0^{R_1} \rho(r_1, \theta) 3r_1^2 \cos \theta dr_1 d\theta = M_0 : \text{integral(\rho)=total mass = } M_0 .
\end{align*}

In order to avoid the clumsy integral condition for \rho , we can introduce the mass M as a variable:

\begin{align*}
M(r, \theta) &= \int_0^{\pi/2} \int_0^{R_1} \rho(r_1, \theta) 3r_1^2 dr_1 d\theta \\
M(r) &= \int_0^{R_1} M(r, \theta) d\theta = \int_0^{\pi/2} \int_0^{R_1} \rho(r_1, \theta) 3r_1^2 dr_1 \cos \theta d\theta \text{ is the mass of the sphere(\textit{r}) and}
\partial_r M(r, \theta) = \rho(r, \theta) 3r^2 \\
\text{For } M(r, \theta) \text{ we impose the boundary condition at } \textit{r= R_1} : \\
M(R_1, \theta) = M_0, \partial_\theta M(R_1, \theta) = 0 \text{ (i.e. density \rho is zero at boundary, and total mass } M_0). \\
\text{So, if we take the diff. equations eqR00, eqR11, eqR22, eqR33, eqR03, eqR41 and replace } \rho(r, \theta) \text{ by } \partial_\theta \rho M(r, \theta), \text{ we have 6 diff.equations in } r, \theta \text{ of degree 2, for the variables } Ai(f(r, \theta)) (\text{metric correction factors=mcf}) \text{ and the mass } M(r, \theta) , \text{ with the highest derivative } \partial_\theta M(r, \theta) \text{ in } M.
According to the Cauchy-Kovalevskaya theorem there exists then a unique solution in a region \( R_i > r > r_i \) within the boundary. Inside the region \( r_i > r > 0 \) we can enforce the vacuum Kerr-spacetime with the trivial solution \( A_i=1, \rho = 0 \), i.e. there is no matter there, \( r_i \) the inner radius.

The Cauchy-Kovalevskaya theorem guarantees the existence of a mathematical solution outside the horizon, but for a physical solution we must have \( r>=0 \) (meaning \( \partial_r M(r, \theta)>0 \)) and \( M(r, \theta)<=0 \) for \( r<= \ r_i \) : the mass must become non-positive at the inner radius.

Therefore, for certain \( \{M_0,R_1\} \) values there will be no physical solution, even for the TOV equation.

4. The solving process for the extended Kerr space-time.

The \( r-\theta \)-slicing algorithm with an Euler-step obeys the iterative procedure with slice step size \( h_i \) in \( r \), and step size \( h_\theta \) in \( \theta \), starting with the \( r \)-boundary at \( r= \ R_i \) (slice \( n=0 \)).

The transition from slice \( n \) to \( n+1 \) proceeds as follows.

At slice \( n \) all variables and 1-derivatives are known from the previous step, 2-derivatives \( \partial_{rr} A_i \), \( \partial_{rr} B_i \) and \( \rho \) are calculated from the 6 equations.

At slice \( n+1 \) the variables and 1-derivatives are calculated by Euler-formula (or Runge-Kutta)

\[
A_{i+1} = A_i + h_\theta \partial_r A_i
\]

\[
\partial_r A_i = \partial_r A_{i+1}
\]

The 2-derivatives \( \partial_{rr} A_i \), \( \partial_{rr} B_i \) and \( \rho \) are again calculated from the 6 significant equations with variables and 1-derivatives inserted from above.

The \( \theta \)-slicing \( r \)-backward algorithm with an Euler-step obeys the iterative procedure with slice step size \( h_\theta \) in \( \theta \) as above for \( r \), starting with \( \theta=0 \), and solves an ordinary differential equation in \( r \) in each \( \theta \)-step .

The boundary condition for the \( r \)-odeq is set at \( r= \ R_i \) (the outer radius) with \( A_i=1, M=M_0 My_0(\theta), \quad \partial_r A_i =0, \quad \partial_r M = 3(r_i)^2 \rho_{bc}, \)

where \( \rho_{bc} \) is the outer boundary value for the density, \( \rho_{bc}=0 \) for the (non-interacting) neutron-gas in a shell-star (black-hole) and \( \rho_{bc}>0, \rho_{bc}=\rho_{equilibrium} \) for the (interacting) neutron fluid in a neutron star. \( My_0(\theta) \) is the mass-form-factor with the condition

\[
\int_0^{\pi/2} My_0(\theta) Cos(\theta)d\theta = 1 , \quad i.e. \quad \text{the overall mass at the outer boundary is } M_0 \ . \quad \text{With the assumption that } My_0 \text{ is simplest possible trigonometric function, this adds a third fundamental parameter } dM=\text{relative-mass-amplitude} \quad \text{to the 2 fundamental TOV-parameters } \{ R_i , M_0 \}. \]

The alternative (dual) \( \theta \)-slicing \( r \)-forward algorithm starts with the boundary condition at \( r= \ r_i(\theta) \)

\[
A_i=1, M=0, \quad \partial_r A_i =0, \partial_r M = 3(r_i)^2 \rho_{bc}, \]

where \( \rho_{bc}=\rho_i \) is the inner boundary value for the density, \( \rho_i \) is approximately the inner (maximum) density \( \rho(r_i) \) from the corresponding TOV-equation, the value must be adapted, so that the resulting total mass is \( M_0 \). For the compact neutron star the inner radius \( r_i(\theta) \) is zero.

It is sufficiently general to assume that \( r_i(\theta) \) is an ellipse with radii \( r_x \) and \( r_y=r_i \), the latter equality arising from the fact that centrifugal distortion acts only in the x-direction (the y-axis being the rotation axis). At the inner boundary the tangential pressure is uniform, so the density is also uniform and equal to the maximum density, \( \rho(\theta)=\rho_i \).
As we see, in addition to the fundamental dual parameters \( \{r_i, \rho_i\} \) corresponding to \( \{R_1, M_0\} \) in the rotation-free TOV-case, in the Kerr-case there is the new fundamental parameter \( \Delta r_i \) (ellipticity) to account for the angular velocity \( \omega \).

The odeqs in \( r \) consist of the 6 significant Einstein equations \( \text{eqR00}, \text{eqR11}, \text{eqR22}, \text{eqR33}, \text{eqR03}, \text{eqR41} \) for the six variables \( A_0(r, \theta), A_1(r, \theta), A_2(r, \theta), A_3(r, \theta), A_4(r, \theta), M(r, \theta) \) with \( \theta = \theta \) and \( \theta \)-derivatives calculated by Euler-step from the preceding q-slice. For \( i = 0 \) i.e. \( \theta = 0 \) the \( \theta \)-derivatives are taken from start values for all variables, which normally represent the corresponding TOV-solution (here only \( A_0(r), A_1(r), M(r) \) are non-trivial and do not depend on \( \theta \)). The odeqs are highly non-linear algebraic differential equations and hard to solve numerically with classical methods for linear odeqs extended by an algebraic equation solver. In the case of a nonlinear odeq-system one uses an Euler or Runge-Kutta method and calculates in each step the highest derivatives with a numerical algebraic equation solver. As an alternative one can use minimization of the least-squares-error in the highest derivatives instead of a numerical algebraic equation solver. Minimization has also the advantage that one can minimize the complete set of Einstein equations plus the 2 additional continuity equations \( \text{eqR41}, \text{eqR42} \) in the error goal function instead of the 6 significant equations, which improves the stability of the solution (e.g. in case of degeneracy).

The numerical error of the algorithm is calculated from \( \max\{\text{abs}(\text{eq}), i=1...n\} \) i.e. the maximum absolute deviation of the equation values from 0. As the equations are homogeneous in \( \{A_i, M\} \) and their derivatives, the error must be scaled somehow. Here, we choose the error of the test-function for \( \{A_i, \rho\} \) \( \text{lorentzhill}(r)\text{flin}(\text{Cos}(\theta), \text{Sin}(\theta)) \) as the error scale for every equation, i.e. the equations will be normalized by this factor. Here \( \text{flin} \) is a function of degree 1, and \( \text{lorentzhill} \) is a parametrized model of a “hill” i.e. function with a finite support area (=0 at both infinities) of the Lorentzian form (step at \( r=1=R \) resp. \( r=1=-R \))

\[
\begin{align*}
\text{bfunc}[r_1, R, dR] &= \frac{1}{1+\exp[(r_1-R)/(dR*R)]} \\
\text{bnfunc}[r_1, R, dR] &= \frac{1}{1+\exp[(r_1+R)/(dR*R)]}
\end{align*}
\]

\( \text{lorentzhill} = \text{bfunc} * \text{bnfunc} \)

\( \text{lorentzhill} \) models the behaviour of \( \rho(r) \), which vanishes at \( r = \pm \infty \).

The actual calculation was carried out in Mathematica using its symbolic and numerical procedures. In the first stage, the Einstein equations were derived from the ansatz for \( g_{\mu\nu} \) from section 2 and simplified automatically. The arising complexity of the equations is such, that it is practically impossible to handle them manually: the Mathematica function \( \text{LeafCount} \), which returns the number of terms in the equation, gives the complexity of \( \text{LeafCount}[\text{eqR00}]=17408, \text{LeafCount}[\text{eqR11}]=27528, \text{LeafCount}[\text{eqR22}]=134929 \) for the first 3 equations. To verify the equations, the TOV equation was derived by symbolic manipulation for \( a=0 a=0 \) from \( \text{eqR00}, \text{eqR11}, \text{eqR22}, \text{eqR41} \).

The power of Mathematica is sufficient to solve the TOV equation with the single procedure \( \text{NDSolve} \). For the full Einstein equations it fails even for the ordinary differential equations (odeq) in \( r \) arising for fixed \( \theta \). It took us a long time to find an algorithm, which could handle the complexity of the equations and solve them in an acceptable time (4 hours) on a PC-desktop and converge in the required region with an acceptable error of around 0.05. For the second numerical stage we tried several slicing algorithms, and the best alternative proved to be the \( \theta \)-slicing \( r \)-backward and \( \theta \)-slicing \( r \)-forward algorithm implemented by hand in Mathematica. The solution of the resulting odeq in each \( r \)-step was far too slow using \( \text{NDSolve} \), so we chose a minimization procedure instead, which had to be carefully tuned to ensure sufficient continuity in \( r \), given the partly random method of minimization.

In each \( r \)-step, there was a fit in \( r \) on the result value list \( \{M(r_i), A_0(r_i),...\} \), which was used
for the next step. The error was continually monitored and the minimization was adapted appropriately. Also, for every star model and parameter set, the TOV solution with \( \omega=0 \) \( a=0 \) was calculated first with the algorithm and compared with the exact TOV solution.

5. The TOV equation as the limit \( \omega>0 \) for the extended Kerr space-time.
In the Schwarzschild spacetime \( \omega=0 \) and \( a=0 \), we have spherical symmetry, no dependence on \( \theta \), then the TOV-equation can be derived from the remaining non-trivial Einstein equations eqR00, eqR11, eqR22, eqR41.

The TOV-equation is in the standard form:

\[
P'(r) = -\left( \frac{GM(r)\rho(r)}{r^2} \right) \left( 1 + \frac{P(r)}{\rho(r)c^2} \right) \left( 1 + \frac{4\pi^3 P(r)}{M(r)c^2} \right) \left( 1 - \frac{2GM(r)}{rc^2} \right)^{-1}
\]

and using \( r_s \)

\[
P'(r) = -\left( \frac{c^2 r_s \rho(r)}{2r^2} \right) \left( 1 + \frac{P(r)}{\rho(r)c^2} \right) \left( 1 + \frac{4\pi^3 P(r)}{M(r)c^2} \right) \left( 1 - \frac{r_s M(r)}{rM_i} \right)^{-1},
\]

where \( M_i \) is the total mass, furthermore

\[4\pi r^2 \rho(r) = M(r), \ P(r) = k_i \rho(r)^7\]

In order to make the variables dimensionless, one introduces ‘sun units’

\[r_{ss} = r_s (\text{sun}) = \frac{2GM_{\text{sun}}}{c^2} = 3km, \ \rho_s = \frac{M_{\text{sun}}}{4\pi^3 r_{ss}^3} = 1.7610^{16} \frac{g}{cm^3}, \ P_s = \rho_c c^2\]

where \( r_s \) Schwarzschild-radius of the sun, \( \rho_s \) the corresponding Schwarzschild-density and \( P_s \) the corresponding Schwarzschild-pressure.

In ‘sun units’ TOV-equation transforms into

\[
P'_1(r_1)r_1^3 (r_1 - M_1(r_1) M_0) = -\frac{1}{2} \left( \frac{M'_1(r_1) M_0}{3} + P_1(r_1) r_1^2 \right) (M_1(r_1) M_0 + 3P_1(r_1) r_1^3)
\]

with the normalized mass \( M1(r1) \) , and \( M_1(R_1) = 1 \),

or

\[
P'_1(r_1)r_1^3 (r_1 - M_1(r_1)) = -\frac{1}{2} \left( \frac{M'(r_1)}{3} \right) + P_1(r_1) r_1^2 \left( M(r_1) + 3P_1(r_1) r_1^3 \right)
\]

where \( M_0 = \frac{M}{M_{\text{sun}}} \), \( M(r) \) is the mass within the radius \( r \), \( M(r_2) = M_0 M_2(r_2) \)

in dimensionless variables \( r_1, \ \rho(r_1) = \frac{M'(r_1)}{3r_1^2} \), \( M, \ P_1 = k_i \rho(r_1)^7\)

and \( R_1 \) is the dimensionless radius of the star.

With the replacement \( P = k_1 \rho r^7 \) for the pressure from the equation of state and

\( \rho = \frac{M_0 M'}{3r^2} \) we obtain a diff. equation for M degree 2 in \( r \) and we impose the boundary condition in \( r=R_1: M(R_1)=M_0, \ M'(R_1)=0 \) for non-interacting Fermi-gas and
for an interacting Fermi-gas: \( M(R) = M_0 \), \( M'(R) = \rho(R) \), \( \rho(R) = \rho_e \), where \( \rho_e \) is the equilibrium density in the minimum of \( V_{\text{nm}} \) and \( P_1'(\rho_e) = 0 \) (here an equivalent boundary condition is \( \rho'(R) = \infty \)).

6. The equation of state and rotation parameters

6.1. The equation of state for an (non-interacting) nucleon gas

Here, \( P = k_1 \rho^2 \) is the equation of state of the star, derived from the thermodynamic Fermi gas equation at \( T=0 \) ([2], chap. 48).

\[
P = -\frac{\partial E}{\partial V} = 8\pi P_0 \left( \frac{x_F^3}{3} \sqrt{1 + x_F^2} - f(x_F) \right)
\]

\[
P_0 = \frac{m c^2}{\lambda_c^3} = \frac{m^4 c^5}{\hbar^3}, \quad \text{where } \lambda_c \text{ is the de-Broglie wavelength of the Fermi gas with particle mass } \ m, \ \frac{\lambda_c}{m c} = \frac{\hbar}{m c}, \ \frac{x_F}{mc} = \frac{\lambda_c}{2} (3\pi)^{1/3} n^{1/3}, \ \text{where } x_F \text{ is the Fermi-angular-momentum, } n \text{ the particle density}
\]

\[
f(x_F) = \int_0^1 dx x^2 \sqrt{1 + x^2}
\]

The resulting approximate equations of state for \( P \) are

\[
P = 8\pi P_0 \begin{pmatrix} x_F^5 \\ 15 \\ x_F^4 \\ 12 \end{pmatrix} = \begin{pmatrix} K_1 \rho^{5/3} \rho << \rho_c \\ K_2 \rho^{3/3} \rho >> \rho_c \end{pmatrix}
\]

valid for the density \( \rho \) and the critical density \( \rho_c \)

\[
\rho_c = \frac{m}{\lambda_c^3} \frac{8\pi}{3}
\]

The full expression for \( P \), including temperature \( T \), is as follows ([4], chap.15). Here, we use dimensionless variables (\( r_1 \) distance unit de-Broglie-wavelength \( \lambda_c \), \( V_1 \) volume unit \( \lambda_c^3 \), \( n_1 \) particle density unit \( 1/\lambda_c^3 \), \( E_1 \) energy unit \( E_0 = \frac{h c}{\lambda_c} = \frac{m c^2}{2\pi} \), inverse thermal energy \( \beta_1 = \frac{E_0}{k T} \), chem. potential \( \mu_1 \) in \( E_0 \)), for the gas model we use the Debye model with the state density \( \frac{D_1(E_1)}{V_1} = \frac{1}{4\pi^{7/2}} \sqrt{E_1} \), maximum energy \( \epsilon_{F1} = \frac{3^{2/3} \pi^{4/3}}{4} n_1^{2/3} \), the resulting particle density is

\[
n_1 = \frac{N_{\text{op}}}{V_1} = \frac{2}{V_1} \int_0^\infty d\omega_1 \frac{D_1(\omega_1)}{1 + \exp(\beta_1(\omega_1 - \mu_1))} = \frac{1}{2\pi^{7/2}} \int_0^\infty d\omega_1 \frac{\sqrt{\omega_1}}{1 + \exp(\beta_1(\omega_1 - \mu_1))}
\]

From this relation the chem. potential \( \mu_1 \) can be calculated, an approximation formula is

\[
\mu_1 = \epsilon_{F1} - \frac{\pi^2}{12 \beta_1^2 \epsilon_{F1}^2} = \mu_1(n_1)
\]

Finally, the resulting pressure (=energy density) \( \rho_1(\beta_1, n_1) \):
Below a 3D-diagram of $p_1(\beta_1, n_1)$ in dimensionless variables for a nucleon gas (m=m$_n$, density $\rho = \frac{E_0 n}{c^2}$ in sun units, $E_0=149.4$MeV) is depicted:

Here $kT$ is in $E_0$ units, and one sees the dependence $P = k1 \rho^\gamma$ except on the left side, when $kT$ reaches the magnitude of 1Gev (T=10^10K).

6.2. The equation of state for an (interacting) nucleon fluid

For the interacting nucleon gas we take into account the nucleon-nucleon-potential in the form of a Saxon-Woods-potential modeled on the experimental data:

$$V_{sw}[r, V_0, r_0, dr_0] = V_0 / (1 + \exp[(r-r_0)/dr_0])$$

$$V_{nn}[r] = V_{sw}[r, V_a, r_a, dr_a] + V_{sw}[r, V_c, r_c, dr_c]$$

where $V_{nn}$ is the nucleon-nucleon-potential with an attractive part $V_{sw}[r, V_a, r_a, dr_a]$ and a repulsive core $V_{sw}[r, V_c, r_c, dr_c]$, the distance $r$ between the nucleons is

$$r = (E_n / \rho)^{1/3},$$

where $E_n = (m, c^2 / (2\pi))^{1/3} = 149.4$MeV $= m_n c^2$ is the nuclear energy scale $m_n$=pion mass = 140MeV, $m_n$=neutron mass = 140MeV.

The pressure of the interacting nucleon fluid becomes then

$$p_1(\beta_1, n_1) = \frac{4\pi^3/2}{3\pi^2} \int d\omega_1 \frac{\omega_1^{3/2}}{1 + \exp(\beta_1(\omega_1 - \mu_1(n_1)))}$$ (17)
\[ P_1(r_i) = c_1 \rho_i(r_i) V_{nn} \left( \frac{E_n}{\rho_i(r_i)} \right)^{1/3} \]  

(18)

The experimental data used here are those from [7]

And the hard-core potential from the lattice calculation Reid93 from [5]

both fitted with a double Saxon-Woods-potential V_{nn} with r(fm), V(MeV).
From the nucleon-nucleon-potential the pressure is calculated taking into account the low-density Fermi-pressure of the nucleons \( K_1 \rho^{5/3} \)

\[
P_{nn}(\rho) = V_{nn} \left( \frac{1}{(\rho)^{1/3}} \right) \rho
\]

\[
P_{fg}(\rho) = K_1 \rho^{5/3} + P_{nn}(\rho)
\]

The total pressure \( P_{fg}(r) \), pressure \( P \) and density \( \rho \) shown in sun-units. This equation-of-state has a minimum at \( \rho = \rho_c = 0.0417 \) and \( P'(\rho) = 1 \) at \( \rho = \rho_m = 0.0544 \).

As the sound velocity \( v = \frac{dP(\rho)}{d\rho}, v > 0 \) and \( v < 1 \) (i.e. subluminal), the admissible density range in the neutron-fluid model is \( \rho_c \leq \rho \leq \rho_m \).

### 6.3. Maximum omega-values in Kerr-space-time

We consider here a rotation model with constant angular velocity \( \omega \). With this model the resulting 4-velocity \( u^\mu \) has the form [11]:

\[
u = (u^0, 0, 0, \omega u^0)
\]

The maximum values for \( \omega \) are calculated from the minimal zeros in omega of the denominator in \( u^0 \) from (9a), minimized over \( r1 \) and \( \theta \) in their respective regions \( r_i \leq r_i \leq R_1 \) and \( 0 \leq \theta \leq \pi/2 \).

The resulting value is \( \omega \leq \frac{1}{2R_1 \sqrt{\alpha_f}} \), where \( \alpha_f \) is the form-factor in the moment of inertia \( I_1 \).

\[
I_1 = \alpha_f \frac{M}{R_i^2}, \quad \alpha_f = 2/3 \text{ for a shell, } \alpha_f = 2/5 \text{ for a sphere.}
\]

With non-vanishing density the actual \( \omega_{\text{max}} \) depends on \( \{A_i, \rho\} \), and has to be calculated from the above expression for \( u^0 \).

### 7. The TOV-equation: a new ansatz

Generally speaking, the parameters of the solution are:

- angular momentum radius \( a (=\alpha_1, =0 \text{ for TOV}) \), the factor in the state equation \( k_1 \), the power in the state equation \( \gamma (=\gamma) \), radius \( R \), mass \( M_0 \), the relative radius uncertainty
The TOV-equation. The TOV-equation is a differential equation in the mass $M(r)$ of degree 2, and is highly non-linear. The dimensionless mass-density relation is $\rho = \frac{\rho'}{3r^2}$. The customary way of solving the TOV equation is to impose the boundary condition at $r=0$ with $M(0)=0$, $M'(0)=0$, and $\rho(0) = \rho_0$, where $\rho_0$ is the maximum central density.

In the new ansatz for the mass $M(r)$, we impose the outer boundary condition at $r=R_1$: for a pure Fermi-gas without interaction: $M(R_1) = M_0$, $M'(R_1) = \rho(R_1) 3 R_1^2$, $\rho(R_1) = 0$; for an interacting Fermi-gas: $M(R_1) = M_0$, $M'(R_1) = \rho(R_1) 3 R_1^2$, $\rho(R_1) = \rho_e$, where $\rho_e$ is the equilibrium density in the minimum of $V_{nn}$ and $P_1'(\rho_e) = 0$ (here an equivalent boundary condition is $\rho'(R_1) = \infty$).

The star parameters mass $M_0$ and radius $R_1$, which enter the outer boundary condition determine completely the solution. In general, there will be an inner radius $r_i > 0$ with the maximum density $\rho_0 = 3 r^2 M'(r_i)$ and $M(r_i)=0$. The corresponding ‘dual’ parameters are the inner radius $r_i$ and the maximum density $\rho_0$. One can show that for $\rho_0 \gg \rho_e$ (where $\rho_e$ is the critical density of the equation of state) there is no solution with a compact star $r_i=0$, i.e. there is a maximum mass $M_c$ for the TOV equation, in case of compact neutron stars $M_c = 3.04 M_{\odot}$ (see below). As we will see, there is in general a solution, if we allow $r_i > 0$ and impose an outer boundary condition at $r=R_1$, as long as $R_1$ is not too close to the Schwarzschild radius $r_s = M_0$ of the star. In the limit $R_1 \to r_s$ there will be no positive zero of $M(r)$, i.e. $r_i < 0$ and the resulting (mathematical) TOV solution will be no physical solution.

But in general, speaking naively, the gravitational collapse of the star is avoided for large masses ($M_0 > M_c$), if it has a shell structure with the inner radius $r_i$ and the outer radius $R_1$. As we will see, this outer boundary condition together with allowing $r_i > 0$ changes dramatically the resulting manifold of physical solutions.

7.1. The TOV-equation: the parametric solution and resulting star types

By setting up a parametric solution of the TOV-equation one gets a map of possible physical solutions, i.e. possible star structures. As parameters one can use either $(M_0, R_1)$ in the outer boundary condition at $r_1=R_1$ or the dual parameter pair $(r_i, \rho_{bc})$ in the inner boundary condition $r_1=r_i$.

Both approaches yield the same results, which are as follows.

**Neutron stars** consist of interacting neutron fluid and are compact stars with $(M_0, R_1) = (0.14, 1.49) ... (3.04, 3.95)$ and the maximum density $0.048 \leq \rho_{bc} \leq 0.0544 = \rho_{bc\text{max}}$ in sun-units, or shell-stars with $\rho_{bc} = \rho_{bc\text{max}}$ and $(M_0, R_1) = (3.04, 3.95) ...(4.91, 4.92)$, neutron star R-M-relation follows approximately a cubic-root-law: $R \sim M^{1/3}$.

**Stellar quasi-black-holes** consist of (almost) non-interacting Fermi-gas of neutrons and are thin shell-stars with $R_i > r_i$, $R_s = r_s$, $\rho_s = \rho_s$, i.e. the shell is close to the Schwarzschild-radius and its outer edge outside the Schwarzschild-horizon with max. density $0.0025 \leq \rho_{bc} \leq 0.042$, and obey an almost linear R-M-relation.
\( (M_0, R_1) = (5.5, 9.1) \ldots (18.8, 20.8) \), independent of \( \rho_{bc} \) for \( \rho_{bc} \geq 0.028 \), with redshift factor around 10 for \( M = 18 \, M_{\odot} \).

**Galactic (supermassive) quasi-black-holes** are very thin shell-stars, which obey the *equation-of-state of a white-dwarf* (i.e. gravitation counterbalanced by Fermi-pressure of electron gas) and have an almost linear R-M-relation with redshift factor 20\ldots 100.

**Neutron stars**

The parametric solution of the TOV-equation has been carried out for the parameters \((\rho_{bc}, r_i)\) at the boundary \( r = r_i \), in the range: density \( 0.02 \leq \rho_{bc} \leq 0.15 \) and inner radius \( 0.01 \leq r_i \leq 15 \), yielding physical solutions for density \( 0.048 \leq \rho_{bc} \leq 0.0544 = \rho_{bc_{\text{max}}} \) and inner radius \( 0.01 \leq r_i \leq 3 \). The TOV-equation is solved for \( M(r) \) and \( \rho(r) \), and a physical solution is a mathematical solution with \( M \geq 0 \) and \( \rho \geq 0 \), \( \rho' \leq 0 \) and subluminal equation-of-state within a certain interval \( r = \{ r_i, r_{02} \} \), which reaches a point, where \( M'(r) = 0 \) and \( \rho(r) = 0 \). The radius \( R_1 \) and the total mass \( M_0 \) is reached at \( M'(R_1) = 0 \), the physical solution ends there.

The validity interval for \( \rho \) is explained by the fact, that the sound velocity \( v_s(\rho) = \frac{\partial P(\rho)}{\partial \rho} \) must be positive and below 1 (subluminal in c-units).

The parametric mapping results in the following dependence for \( M_0(r_i, \rho_{bc}) \), \( R_1(r_i, \rho_{bc}) \) (\( r_i, \rho_{bc}, M_0, R_1 \) in sun-units):

For \( r_i = 0 \) the mapping describes the compact neutron stars, resulting in \( R_1(M_0) \) function:
The R-M-relation follows approximately a cubic-root-law: $R \propto M^{1/3}$, with a range of $(M_0, R_1) = (0.14, 1.49) \ldots (3.04, 3.95)$, i.e. the resulting maximum compact mass is $M_{\text{max}} = 3.04 M_{\odot}$.

For $M_0 > M_{\text{max}}$, the function $R_1(r_i = \text{const}, \rho_{bc})$ is flat or slightly decreasing with $\rho_{bc}$, so one expects the stable configuration to be the one with maximum $\rho_{bc} = \rho_{bc\text{max}}$:

The maximum mass for a repulsive-hardcore-model for the equation-of-state DD2 [10] is $2.42 M_{\odot}$, from our mapping we have the maximum compact neutron star mass of $M_{\text{max}} = 3.04 M_{\odot}$.

The actual theoretical limit for neutron star core density is $\rho_{\text{max}} = 3.5 \times 10^{15} \text{ g/cm}^3 = 0.199$ in sun-units [8,9].

The limit for $\rho_{bc}$ reached in our mapping is only $1/4$ of this $\rho_{bc} = \rho_{bc\text{max}} = 0.0544$, due to the subluminal-sound-condition and the use of an (attractive) nucleon-nucleon-potential for the nucleon-fluid instead of a pure repulsive-hardcore-model.
The classical argument for the collapse of a neutron star to a black-hole for $\rho_{bc} > \rho_{\text{max}}$, dating back to Oppenheimer [2], is invalidated here by the simple introduction of shell-star models, where $r_i > 0$, and therefore there is no mass at the center, which means physically, there is only a very diluted nucleon gas there.

**Stellar quasi-black-holes**

The parameter range of the mapping is: density $0.025 \leq \rho_{bc} \leq 0.0417 = \rho_{\text{oc}}$ and inner radius $0.01 \leq r_i \leq 90$, where $\rho_{\text{oc}}$ is equilibrium value of the nucleon-nucleon-potentials with $P_n'(\rho_{\text{oc}}) = 0$, the transition point from the nucleon-fluid to the nucleon-gas phase. The mapping gives the $M_0 - R_1$-range of $(M_0, R_1) = (5.5, 9.1) ... (18.8, 20.8)$. The underlying equation-of-state is the Fermi-gas of nucleons with the low-density limit of $P(\rho) = K_1 \rho^{5/3}$. 

As the resulting $M_0(r_i, \rho_{bc})$, $R_1(r_i, \rho_{bc})$ are practically independent of $\rho_{bc}$, we see that the M-R-relation is given by the $M_0, R_1$ values for $\rho_{bc} = 0.0417 = \rho_{\text{oc}}$. The resulting R-M-relation is practically linear and has a maximum mass value of $M_{\text{max}} = 18.8$. 

\[ \text{As the resulting } M_0(r_i, \rho_{bc}) , R_1(r_i, \rho_{bc}) \text{ are practically independent of } \rho_{bc} , \text{ we see that the M-R-relation is given by the } M_0, R_1 \text{ values for } \rho_{bc} = 0.0417 = \rho_{\text{oc}} . \text{ The resulting R-M-relation is practically linear and has a maximum mass value of } M_{\text{max}} = 18.8 . \]
And the corresponding relative shell thickness \( dR_{\text{rel}} = \frac{dR}{M} \) is

\[
M_{\text{bc}} = 0.0417558
\]

and the relative Schwarzschild-distance \( dR_{\text{srel}} = \frac{(R-M)}{M} \) is

\[
M_{\text{bc}} = 0.0417558
\]

The inverse of \( dR_{\text{srel}} \) gives roughly the light attenuation factor of \([1.7...13.9]\). Taken the attenuation factor and the small relative shell thickness of around 0.042, these stellar quasi-black-holes have approximately the properties expected of a genuine black-hole, when measured from a distance \( r >>> R_1 \). Furthermore, the phase space volume of a thin spherical shell is proportional to its surface \( A \), which approximates the Bekenstein black-hole entropy formula \( S = (k_B/L_P^2)A/4 \).

**Galactic (supermassive) quasi-black-holes**

The mean density of a black-hole scales with its radius \( R \) like \( \rho(R) = \frac{M}{V} = \frac{R}{(4/3)\pi R^3} = \frac{3}{4\pi R^2} \)

i.e. for supermassive black-hole with \( M=10^6 M_{\text{sun}} \) we have \( \rho = 10^{-12} \) in sun units (su).

In the following we use the abbreviation \( M_{\text{sun}} = 10^6 M_{\text{sun}} \).

The density scale of a white-dwarf star is \( 10^6 \text{g/cm}^3 = 5.7 \times 10^{-11} \text{ su} \) [2]. Therefore it is plausible to try a parametric mapping with the white-dwarf equation-of-state, where the underlying Fermi-pressure is that of an electron gas instead of a nucleon gas, i.e. equation-of-state

\[ P_f(n_f) = k_f \rho_f(n_f)^\gamma \]

for a pure Fermi gas, \( \gamma = 5/3 \) if the density is below the critical density \( \rho_c \).

The results for \( M_0(r_r \rho_{bc}) \), \( R_1(r_r \rho_{bc}) \) are shown below:
From this result one can draw several consequences: first, the actual density is around $10^{-12}$, that is well below the critical density for a white-dwarf of $\rho_c = 0.91 \times 10^6 \text{g/cm}^3 = 5.17 \times 10^{-11} \text{su}$: $\gamma=5/3$ in the equation-of-state is justified. Second, the viable solutions lie to the left of a "ridge" reaching up to masses around 30 MM$_{\text{sun}}$. Third, a stable solution for a fixed mass will have the highest possible maximum density $\rho_{bc}$ and that will lie on the "ridge". So one can calculate the R-M-relation following the "ridge".

The resulting R-M-relation is as follows:
The R-M-relation is almost linear, as expected, and goes up to 50MM$_{\text{Sun}}$. $dR_{\text{rel}} = (R_1 - r_i)/M_0$ is the relative thickness, and shows, that the shells are very thin indeed, with a minimum of 0.001. The fourth diagram shows the relative Schwarzschild-distance $dR_{\text{srel}} = (R_1 - M_0)/M_0$, which has a minimum at
\[ \{ M_0, dR_{\text{srel}} \} = \{ 7, 0.00142857 \}, \]
so that its reciprocal value (approximate light attenuation factor) is around 700. So the overall result is, that the supermassive quasi-black-holes become ever thinner shells, while the distance from the Schwarzschild-horizon is increasing.

7.2. The TOV-equation: a case study for typical star types

In the nearly-rotation-free case the solution of the TOV-equation was calculated for 4 models (sun units with $r_s =$ Schwarzschild radius

\[ r_{ss} = r_s (\text{sun}) = \frac{2GM_{\text{sun}}}{c^2}, \rho_s = \frac{M_{\text{sun}}}{4\pi r^3/3}, P_s = \rho_s c^2 \]:

\[ r_{ss} = 3 \text{km}, \rho_s = 1.76 \times 10^{16} \text{ g/cm}^3, M_{\text{sun}} = 3 \times 10^{30} \text{ kg}, \]
- average compact neutron star with mass $M_0=0.932M_{\text{sun}}$, radius $R_1=2.767r_{ss}$
- maximum mass neutron shell-star $M_0=4.91, R=4.926$
- white dwarf with $M_0=0.6M_{\text{sun}}$, radius $R_1=3000r_{ss}$
- stellar black hole with $M_0=15.69M_{\text{sun}}$, radius $R_1=17.89r_{ss}$, inner radius $r_i=17.89r_{ss}$
- galactic black hole with $M_0=4.367 \times 10^6M_{\text{sun}}$, $R_1=4.380 \times 10^6r_{ss}$, $r_i=4.356 \times 10^6r_{ss}$

Compact neutron star

parameters= \{ k1=0.40, gam=5/3, M0=0.932, R1=2.767, rhobc=0.0456, ri=0.01 \};
The mean density is here $\rho_{\text{mean}} = \frac{M_0}{R_1} = 0.04447$.

The critical density of the neutron Fermi gas with neutron mass $m_n$ is $\rho_{cn} = \frac{m_n^4 c^3}{3\pi^2 \hbar^3} = 0.35$ (see [2]) , so the low-density approximation with $\gamma=5/3$ can be used.

Results TOV:

Maximum mass neutron shell-star

parameters = { $k_1=0.40, \gamma=5/3, M_0=4.91, R_1=4.926, \rho_{bc}=0.0544, r_i=3$};

The mean density is here $\rho_{\text{mean}} = 0.0530$.

Results TOV:
Stellar quasi-black-hole

parameters = { k1=0.40, gam=5/3, M0=15.69, R1=17.89, rhobc=0.0359, ri=17.);

The mean density is here $\rho_{\text{mean}} = 0.0194$.

The resulting rho and M are:

Here the radius $R_1$ is reached, when $M'(R_1)=0$, i.e. $\rho(R_1)=0$.

White-dwarf star

parameters = { k1=1.43*10^6, gam=5/3, M0=0.6, R1=3000, rhobc=2.02*10^{-11}, ri=0.};

The underlying state equation is that of a small-momentum electron Fermi-gas with the critical density [2] $\rho_{\text{cw}} = \frac{m_e m_n^2 c^3}{3 \pi^2 \hbar^3} = 0.517*10^{-10}\text{su}$.

The mean density is here $\rho_{\text{mean}} = 2.22*10^{-11}$, the maximum deviation of $\rho$ is $\Delta_{\text{max}}\rho = 0.21*10^{-11}$, so the density is practically constant, as expected.

The solution of the TOV-equation becomes

Here the radius $R_1$ is reached, when $M'(R_1)=0$, i.e. $\rho(R_1)=0$. 
**Galactic quasi-black-hole**

parameters= \{ k11(\gamma=5/3)=0.0243*10^6, k12(\gamma=4/3)=0.067*10^4, M0=4.367*10^6, R1=4.380*10^6, \rho_{bc}=4.934*10^{-12}, ri=4.356*10^6\};

TOV equation was solved with an exterior boundary condition \( r02=R1 (M(r02)=M0, \ M'(r02)=0) \), which is equivalent to the interior boundary condition \( r01=ri (M(r01)=0, \rho (r01)=\rho_{bc}) \), and with the full Fermi-gas equation-of-state instead of the simple power law \( P(\rho) = K_1 \rho^{\gamma_1} \).

The mean density is here \( \rho_{mean} = 3.16*10^{-12} \).

The “naive” mean density is here \( \rho_{\text{mean}} = \frac{M_0}{R_1^3} = 3.16*10^{-12} \), i.e. by a factor 10 lower than the mean density of white dwarf. Therefore, despite its huge mass, the galactic black hole can be described by the state equation of a small-momentum (undercritical) Fermi electron gas with the relative density \( x_f = \frac{\rho}{\rho_{ch}} = 0.0612 \) much smaller than that for the white dwarf.

TOV-solution for \( \rho \) (in \( 10^{-12} \) units), \( M \) (in \( 10^6 \) units) in \( r \) (in \( 10^6 \) units), is:

Here there is an internal “hole” with a radius \( r_i = 4.356*10^6 \), maximum \( \rho = 4.934*10^{-12} \) at \( r_i \). The inner radius \( r_i \) lies a little below the Schwarzschild-radius \( r_s = M0 \). The relative shell thickness \( dR_{rel} = (R_1 - r_i)/M0 = 0.00551 \), the relative Schwarzschild-distance \( dR_{srel} = (R_1 - M0)/M0 = 0.00290 \), the light attenuation factor is roughly \( 1/dR_{srel} = 344 \).

Furthermore, \( r_i \) is little sensitive to the temperature up to \( T = 10^7 K \). As for a stellar black hole, when \( R \) converges to \( r_s = M0 \), so does the inner radius \( r_i \), and there is no physical solution (with positive \( \rho \) and \( M \)) for a boundary within the horizon.

**8. The three star models for Kerr-space-time with mass and rotation**

The calculation of Kerr-space-time with mass and rotation was carried out for 3 star models:

- a typical compact neutron star with mass around 1 solar mass
-a presumably typical stellar quasi-black-hole with a mass around 15 solar masses
-a comparatively small galactic quasi-black-hole modelled on the central black-hole in the Milky Way with a mass of around 4 million solar masses

The angular velocity $\omega$ was chosen at $0.65 \omega_{\text{max}}$, i.e. about 2/3 of the maximum value.

We are using the so called “sun units” $\text{sun Schwarzschild-radius } r_s = r_1 (\text{sun}) = 3 \text{ km}$, sun mass $M_s = M (\text{sun}) = 3 \times 10^{30} \text{ kg}$, sun Schwarzschild-density $\rho_s = \frac{M_s}{4\pi r_s^3} = 1.76 \times 10^{16} \text{ g/cm}^3$

sun Schwarzschild-pressure $P_s = \rho_s c^2 = 1.58 \times 10^{35} \text{ J/m}^3$ for radius $r$, mass $M$, density $\rho$, and pressure $P_i$, respectively.

The mass element here is $M_i(r, \theta) = M_1 (r, \theta)$ and the ring mass $M_i(r)$ is the differential mass of the $\theta$-beam $d\theta M_1 (r) = d\theta \int_0^r M_1 (r, \theta) dr, \text{ the density } \rho$ is

$4\pi \cos(\theta) \rho(r, \theta) dr d\theta = \partial_\theta \partial_r M_1 (r, \theta)$. 

As for the result values, dthrel is the maximum relative angular deviation (in $\theta$), and error (relative to the test function error): wavefront error is the median (on lattice) algorithm error, the spline fit and the Fourier fit error is the error of the respective fit of the discrete solution on the lattice.

We are using here the $\theta$-slicing $r$-forward algorithm. For each of the star models a verification step is run first with the angular velocity $w=0$, the result must be the same as in the corresponding TOV-equation. Then a parameter study is made for different ellipticities $\Delta r_i$ at the inner boundary condition in order to find $\Delta r_i$ with a minimal mean energy density: this is the stable solution of the Kerr-Einstein equations.

The parameter denomination is:

$a=\alpha_1$ with $a = \frac{J}{Mc}$ the angular momentum radius (amr) of the Kerr model,

$\omega=\omega_1$ is the angular velocity, $R_1 = R_1=r02$ is the outer radius, $M_0=M0$ is the total mass, $r1$ radius variable, $\theta$ angle variable,

$M(r,\theta)=M1(r1,\theta)$ is the mass function, $A0(r1,\theta)$... $A4(r1,\theta)$ Kerr correction-factor functions, $\rho(r,\theta)=$rho$(r1,\theta)$ is the density function,

$k1$ is the parameter in the approximate Fermi-gas equation-of-state $P_1(r_i) = k_i \rho_i (r_i)^{\gamma}, \gamma=\gamma_1, \gamma_1=5/3, \gamma_2=4/3$, infac is the moment of inertia factor $f_i$, epis is the singularity cancellation parameter with limit(epsi)=0 introduced to improve the numerical stability in singularities

$r_i=r_iact$ is the polar inner radius $R_i$,

$\Delta r$; the ellipticity is the difference between the polar $R_y$ and the equatorial inner radius $R_x$, $R_x=R_y-\Delta r_i$

rilow is the minimal radius $r1$ reached in the solution

$\rho_{bc}=$ rhobcx is the boundary condition density

**Typical compact neutron star**

The underlying star model here is a compact ($r=0$) neutron star of neutron liquid (i.e. strongly interacting neutrons), mass $M_0=0.93$ sun-masses, radius $R_1=2.76$ sun-Schwarzschild-radii (=8.28km).

parameters= {alpha1=0.74,omega1=0.188,k1=0.4,R1=2.7602,gam=5/3,gam1=5/3,gam2=4/3, M0=0.932,dr02rel=0.33,infac=2/5,epsi=0.05, rilow=0.05, rhobcx=0.03396,riact=0.01}
The *r-forward* solution is first calculated with the lattice \(n_x=32, n_y=16\) for the rotation-free TOV-case with a corrected TOV-solution (rho-factor 1.05, R1-factor 1.17) as the initial function. The solution for the Kerr-case starts with this corrected TOV-solution and yields the values:

- radius(\(\theta\)) \(r_0^{2e}=[2.92...2.73]\), mean=2.82, max. rel. angular deviation \(\delta \theta_{rel}=0.063\)
- ring-mass(\(\theta\)) \(M_0^{2e}=[1.12...0.079]\), mean=0.91, \(\delta \theta_{rel}=0.179\),
- \(\delta \theta_{rel}(M1)=0.15, \delta \theta_{rel}(\rho)=0.099, \delta \theta_{rel}(A0)=0.019\),
- total mass \(M_0^{2eff}=0.932\)
- error: \(\text{med(err)}=0.0015(0.054)\) wavefront, \(=0.045(6.72)\) spline fit \(=0.020(6.74)\) Fourier fit
- mean energy density=0.0340

\[
\text{density over } x=\text{radius } r_1, y=\text{angle } \theta
\]

The density distribution is similar to the TOV-case with a decrease in \(\theta\)-direction of \(\delta \theta_{rel}(\rho)=0.099\).
The rotation results in the flattening in the polar direction of \( \Delta \theta_{\text{rel}} = 0.063 \). The neutron star behaves like a fluid because of its “viscosity”, that is, its nuclear interaction and becomes “pumpkin-like”.

**Typical stellar quasi-black-hole**

The star model here is a shell-star \((r > \text{Schwarzschild-radius})\) with mass \( M_0 = 15.69 \) sun-masses, radius \( R_1 = 17.89 \) sun-Schwarzschild-radii (53.67km). The ellipticity \( \Delta r_i \) is at first a free parameter and it is fixed by the requirement of minimal mean energy density to \( \Delta r_i = 0.3 = 0.1677 R_1 \).

Parameters: \( \{ \alpha_1 = 8.7, \omega_1 = 0.0126, k_1 = 0.4, R_1 = 17.89, \gamma = 5/3, \gamma_1 = 5/3, \gamma_2 = 4/3, M_0 = 15.69, \text{infac} = 2/3, \epsilon_\theta = 0.1, \text{rilow} = 15.9, \rho_{\text{bcox}} = 0.036, \text{riact} = 17.004 \} \)

The \( r \)-forward solution is first calculated with the lattice \( \{ n_x = 32, n_y = 16 \} \) for the rotation-free TOV-case as the initial function. The result is correct in the first iteration, so there is no rho-correction for the initial function.

Then a case study with the parameter ellipticity \( \Delta r_i \) is carried out in order to find the minimal mean energy density.

The case study yields a minimum at \( \Delta r_i = 0.3 \) (cigar-like inner boundary), with a mean energy density \( \rho_{\text{bcx}} = 0.0371 \).

The ensuing \( r \)-forward solution with this ellipticity and maximum density \( \rho_{\text{bcx}} = 0.0371 \) at the inner boundary yields the values:

- \( \text{radius}(\theta) \) \( r^{02e} = (17.61...18.06) \), mean = 17.85, max. rel. angular deviation \( \Delta \theta_{\text{rel}} = 0.026 \)
- \( \text{ring-mass}(\theta) \) \( M^{02e} = (15.70...16.03) \), mean = 15.80, \( \Delta \theta_{\text{rel}} = 0.038 \)
- \( \text{inner radius} \) \( r_i = (16.70...17.0) \), mean = 16.85, \( \Delta \theta_{\text{rel}} = 0.017 \)
- \( \Delta \theta_{\text{rel}}(M1) = 0.022, \Delta \theta_{\text{rel}}(\rho) = 0.064, \Delta \theta_{\text{rel}}(A0) = 0.345 \)
- Total mass \( M^{02e \text{eff}} = 15.74 \)
error: med(err)=0.012(0.26) wavefront, =0.019(0.58) spline fit =0.017(0.060) Fourier fit
mean energy density=0.0155

density over x=radius r1, y=angle th

density over angle th at r1=17.12
The density distribution increases in \( \theta \)-direction with the relative span of
\( d\theta_{rel}(\rho)=0.064 \).

(ring) mass profile for \( \theta=0.1 \) (equatorial) and \( \theta=1.4708 \) (polar)
The physical mass distribution ends at $M_1'(r_{02e})=0$, i.e. at the end of the plateau, where the density becomes $\rho=0$.

\[
\begin{array}{c}
\text{effective radius } r_{02e} \text{ over angle } \theta \\
\end{array}
\]

The ring mass has a sharp increase at about the half angle (45°) because of the larger density and radius at the poles.

A remarkable result, distinct from the case of the neutron star, is the shape with rotation. The stellar quasi-black-hole behaves like a ball of neutron gas (negligible interaction) and decreases slightly its equatorial radius, so that, speaking naively, the increased gravitation counteracts the centrifugal force, the shell-star becomes “cigar-like”, with the shell thickness approximately constant.

The \textit{r-backward} solution with $\Delta r_{02e}=0.4$ and $\Delta M_{02e}=0.3$ yields similar results:

- radius($\theta$) $r_{02e}=\{17.60...18.03\}$, mean=17.81, max. rel. angular deviation $\Delta \theta_{\text{rel}}=0.036$
- ring-mass($\theta$) $M_{02e}=\{15.74...16.08\}$, mean=15.82, $\Delta \theta_{\text{rel}}=0.0362$
- inner radius $r_i=\{16.59...17.08\}$, mean=16.85, $\Delta \theta_{\text{rel}}=0.030$

The stellar quasi-black-hole has all its mass concentrated within a thin shell (d$R_1$=$r_{02e}$-$r_i=1.01$) which is situated outside its Schwarzschild-radius $M_0=15.69$, where the minimum distance from the horizon is $\min(r_{02e})-M_0=1.91$, therefore the light energy loss is approximately $M_0/\min(r_{02e})=0.891$ and the attenuation factor $1/(1-M_0/\min(r_{02e}))=9.21$, it means that visible green light of 0.514$\mu$m is shifted to 4.73$\mu$m into middle-range infrared.

\textbf{Galactic quasi-black-hole}

This is modelled (approximately) on the central black-hole in the Milky Way with mass $M_0=4.36$ mega-sun-masses, radius $R_1=4.38$ mega-sun-Schwarzschild-radii ($13.14 \times 10^6\text{km}$).

In order to maintain numerical performance, we are using for mass and distance $10^6$ (mega) units $10^6 M_\odot$ and $10^6 r_{ss}$ and for density $10^{-12}$ (mega$^{-2}$) unit $10^{-12} \rho_\odot$. 
Like in the case of the stellar quasi-black-hole, the ellipticity $\Delta r_i$ is at first a free parameter and it is fixed by the requirement of minimal mean energy density to $\Delta r_i=0.01725$ mega $=0.00394\, R_1$.

parameters= {alpha1=0.670047,omega1=0.05239,k1=0.0243,k2=0.067,R1=4.38,gam=5/3, gam1=5/3,gam2=4/3,M0=4.36731,infac=2/3,eps1=0.0024, rilow=4.3,rhocbx=4.50374,riact=4.357}

The r-forward solution is first calculated with the lattice \( n_x=32, n_y=16 \) for the rotation-free TOV-case as the initial function. The result is correct in the first iteration, so there is no rho-correction for the initial function, the initial density $\rho_{bc}=5.038$.

Then a case study with the parameter ellipticity $\Delta r_i$ is carried out in order to find the minimal mean energy density.

The case study yields a minimum at $\Delta r_i=0.01725$ (cigar-like inner boundary), with a mean energy density $=2.004$.

The r-forward solution with this ellipticity and maximum density $\rho_{bc}=4.560$ at the inner boundary and rho-correction factor $\rho_{fact}=0.9985$ yields the values:

radius($\theta$) $r_{02e}=\{4.365...4.387\}$, mean $=4.378$, max. rel. angular deviation $d\theta_{rel}=0.00491$

ring-mass($\theta$) $M_{02e}=\{4.396...4.685\}$, mean $=4.491$, $d\theta_{rel}=0.217$,

inner radius $r_i=\{4.339...4.357\}$, mean $=4.348$, $d\theta_{rel}=0.00388$

d$\theta_{rel}(M1)=0.0400, d\theta_{rel}(\rho)=0.155, d\theta_{rel}(A0)=0.700$

total mass $M_{02eff}=4.3679$

error: med(err)$=0.0192(1.845)$ wavefront, $=0.544$ spline fit $=0.0666$ Fourier fit mean energy density $=2.004$.

density over x=radius $r_1$, y=angle $\theta$

density over angle $\theta$ at $r_1=4.353$

The density distribution increases in $\theta$-direction with the relative span of $d\theta_{rel}(\rho)=0.155$. 
(ring) mass profile for $\theta=0.1$ (equatorial) and $\theta=1.4708$ (polar)

As in the case of the stellar quasi-black-hole, the polar radius is larger than the equatorial radius, so the outer shape and the inner shape are both cigar-like, but the outer radius has a sharp increase at about a third of the full angle ($30^\circ$). The shell thickness is almost the same at the equator and at the poles.
(ring) mass $M_1(r_{02e}, \theta)$ over angle $\theta$

As in the case of the stellar quasi-black-hole, the ring mass has a sharp increase at about a third of the full angle ($30^\circ$) because of the larger density and radius at the poles.

The $r$-backward solution with $\Delta r_{02e}=0.0230$ and $\Delta M_{02e}=-0.007$ with minimal mean energy density over $\Delta r_{02e}$ and $\Delta M_{02e}$ yields similar results:

radius ($\theta$) $r_{02e}=\{4.356...4.378\}$, mean=4.367, max. rel. angular deviation $d\theta_{rel}=0.00517$

ring-mass ($\theta$) $M_{02e}=\{4.172...4.757\}$, mean=4.552, $d\theta_{rel}=0.129$,

inner radius $r_i=\{4.313...4.336\}$, mean=4.324, $d\theta_{rel}=0.00526$.

The galactic quasi-black-hole is a shell object with a thin mass shell ($\Delta R=0.057$) situated close above its outer horizon $r_+=4.262$.

The maximum distance from the horizon is max($r_{02e})- r_+=0.125$, therefore the minimal light energy attenuation is roughly $4.262/0.125=34$, it means that visible green light of 0.514 $\mu$m is shifted to 17 $\mu$m into far-infrared.

9. Conclusions

We introduce in chap. 6 an eos for the nucleon-fluid in the density range $\rho_c<\rho<\rho_m$, where

$\rho_c=0.0417 \rho_s$ and $\rho_m=0.0544 \rho_s$ (sun units $r_{ss}=3km$, $\rho_s=1.76 \times 10^{16} g/cm^3$), which is based on measurement data for the nucleon-nucleon-potential. This suggests that there is a phase transition at $\rho=\rho_c$ from the (interacting) nucleon fluid to the (weakly interacting) nucleon Fermi-gas.

Based on these 2 eos’s the results for the TOV-equation in chap. 7 are as follows.

Neutron stars obey the nucleon fluid eos and there are compact neutron stars in the range $(M_0, R_1)=(0.14 M_{sun}, 1.49 r_{ss})...(3.04 M_{sun}, 3.95 r_{ss})$, the R-M-relation follows approximately a cubic-root-law: $R\sim M^{1/3}$.

Neutron shell-stars exist in the range $(M_0, R_1)= (3.04 M_{sun}, 3.95 r_{ss})...(4.91 M_{sun}, 4.92 r_{ss})$.

Stellar quasi-black-holes exist in the range of $(M_0, R_1)= (5.5 M_{sun}, 9.1 r_{ss})...(18.8 M_{sun}, 20.8 r_{ss})$.

The underlying equation-of-state is the Fermi-gas of nucleons with the eos

$P(\rho) = K_i \rho^{5/3}$. The resulting R-M-relation is practically linear and has a maximum mass value of $M_{max}=18.8 M_{sun}$. The light attenuation factor (redshift) is roughly $1.7...13.9$. Taken the redshift and the small relative shell thickness of around 0.042, these stellar quasi-black-holes have approximately the properties expected of a genuine black-hole, when measured from a distance $r>>(R_1$).

Furthermore, the phase space volume of a thin spherical shell is proportional to its surface $A$, which approximates the Bekenstein black-hole entropy formula $S=(k_B/\hbar^2)A/4$.

The galactic (supermassive) quasi-black-holes have the density scale and the eos of a white-dwarf-star, i.e. of an electron Fermi-gas. The R-M-relation is almost linear and goes from $1M_{sun}$ up to $50M_{sun}$ ($MM_{sun}=10^6M_{sun}$, $Mr_{ss}=10^6 r_{ss}$). $dR_{rel}=(R_1-r_1)/M_0$ is the relative
thickness, and shows, that the shells are very thin indeed, with a minimum of 0.001. The relative Schwarzschild-distance \( dR_{srel} = (R_1-M_0)/M_0 \) has a minimum at \( \{ M_0, dR_{srel} \} = \{ 7. MM_{sun}, 0.00142857 \} \), the redshift is around 700. So the overall result is, that the supermassive quasi-black-holes become ever thinner shells, while the distance from the Schwarzschild-horizon is increasing.

In chap. 8 we present numerical results for rotating stars of the 3 types compact neutron star, stellar quasi-black-hole and galactic quasi-black-hole.

The angular velocity \( \omega \) was chosen at \( \omega = 0.65 \omega_{max} \), i.e. about 2/3 of the maximum.

The compact neutron star with \( M_0 = 0.932 MM_{sun} \), \( R_{1y} = 2.73 r_{ss} = 8.19 \text{km} \), \( R_{1x} = 2.92 r_{ss} \), has the relative ellipticity of \( dthrel = 0.099 \). The neutron star behaves like a fluid because of its “viscosity”, that is, its nuclear interaction, and becomes “pumpkin-like”.

The stellar quasi-black-hole with \( M_0 = 15.74 MM_{sun} \), \( R_{1mean} = 17.85 r_{ss} \), has maximum density \( \rho_{bc} = 0.0371 \rho_s \), outer radii \( R_{1y} = 18.06 r_{ss} = 54.18 \text{km} \), \( R_{1x} = 17.61 r_{ss} \), inner radii \( r_{iy} = 17.0 r_{ss} \), \( r_{ix} = 16.7 r_{ss} \), outer rel. ellipticity \( dthrel = 0.026 \). The ring-mass (the differential mass of the \( \theta \)-beam) has a sharp increase at half-angle \( \theta = 45^\circ \). The redshift is 9.21.

The stellar quasi-black-hole behaves like a ball of neutron gas (negligible interaction) and decreases slightly its equatorial radius, so that, speaking naively, the increased gravitation counteracts the centrifugal force, the shell-star becomes “cigar-like”, with the shell thickness approximately constant.

The galactic quasi-black-hole is modelled (approximately) on the central black-hole in the Milky Way with mass \( M_0 = 4.368 MM_{sun} \) \( (MM_{sun} = 10^6 M_{sun}, Mr_{ss} = 10^6 r_{ss}) \), radius \( R_1 = 4.38 Mr_{ss} (=13.14 \times 10^6 \text{km}) \).

It has maximum density \( \rho_{bc} = 5.038 \times 10^{-12} \rho_s \), outer radii \( R_{1y} = 4.387 Mr_{ss} \), \( R_{1x} = 4.365 Mr_{ss} \), inner radii \( r_{iy} = 4.357 Mr_{ss} \), \( r_{ix} = 4.339 Mr_{ss} \), outer rel. ellipticity \( dthrel = 0.00491 \).

As in the case of the stellar quasi-black-hole, the ring mass has a sharp increase at about a third of the full angle \( (30^\circ) \). The redshift is roughly 34. The galactic quasi-black-hole is a shell object with a thin mass shell \( (\Delta R = 0.057 Mr_{ss}) \) situated close above its outer horizon \( r_s = 4.26 Mr_{ss} \). As in the case of the stellar quasi-black-hole, the polar radius is larger than the equatorial radius, so the outer shape and the inner shape are both cigar-like, but the outer radius has a sharp increase at about a third of the full angle \( (30^\circ) \).

The overall result is, that the introduction of numerical shell-star solutions of the TOV- and Kerr-Einstein-equations creates quasi-black-hole star models, which mimic closely the behaviour of abstract black holes and satisfy the Bekenstein entropy formula, but have finite redshifts and escape velocity \( v < c \), no singularity, no information loss paradox, and are classical objects, which need no recourse to quantum gravity to explain their behaviour.
References