On values of arithmetical functions at factorials I

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1. The Smarandache function is a characterization of factorials, since $S(k!) = k$, and is connected to values of other arithmetical functions at factorials. Indeed, the equation

$$S(x) = k \quad (k \geq 1 \text{ given})$$

has $d(k!) - d((k - 1)!)$ solutions, where $d(n)$ denotes the number of divisors of $n$. This follows from $\{x : S(x) = k\} = \{x : x|k!, \ x \nmid (k - 1)!\}$. Thus, equation (1) always has at least a solution, if $d(k!) > d((k - 1)!)$ for $k \geq 2$. In what follows, we shall prove this inequality, and in fact we will consider the arithmetical functions $\varphi, \sigma, d, \omega, \Omega$ at factorials. Here $\varphi(n)$ = Euler’s arithmetical function, $\sigma(n)$ = sum of divisors of $n$, $\omega(n)$ = number of distinct prime factors of $n$, $\Omega(n)$ = number of total divisors of $n$. As it is well known, we have $\varphi(1) = d(1) = 1$, while $\omega(1) = \Omega(1) = 0$, and for $1 < \prod_{i=1}^{r} p_i^{a_i}$ ($a_i \geq 1$, $p_i$ distinct primes) one has

$$\varphi(n) = n \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i} \right),$$

$$\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{a_i+1} - 1}{p_i - 1},$$

$$\omega(n) = r,$$

$$\Omega(n) = \sum_{i=1}^{r} a_i.$$
\[ d(n) = \prod_{i=1}^{r}(a_i + 1). \] (2)

The functions \( \varphi, \sigma, d \) are multiplicative, \( \omega \) is additive, while \( \Omega \) is totally additive, i.e. \( \varphi, \sigma, d \) satisfy the functional equation \( f(mn) = f(m)f(n) \) for \( (m, n) = 1 \), while \( \omega, \Omega \) satisfy the equation \( g(mn) = g(m) + g(n) \) for \( (m, n) = 1 \) in case of \( \omega \), and for all \( m, n \) is case of \( \Omega \) (see [1]).

2. Let \( m = \prod_{i=1}^{r} p_i^{\alpha_i}, \; n = \prod_{i=1}^{r} p_i^{\beta_i} \) (\( \alpha_i, \beta_i \geq 0 \)) be the canonical factorizations of \( m \) and \( n \). (Here some \( \alpha_i \) or \( \beta_i \) can take the values \( 0 \), too). Then

\[ d(mn) = \prod_{i=1}^{r}(\alpha_i + \beta_i + 1) \geq \prod_{i=1}^{r}(\beta_i + 1) \]

with equality only if \( \alpha_i = 0 \) for all \( i \). Thus:

\[ d(mn) \geq d(n) \] (3)

for all \( m, n \), with equality only for \( m = 1 \).

Since \( \prod_{i=1}^{r}(\alpha_i + \beta_i + 1) \leq \prod_{i=1}^{r}(\alpha_i + 1) \prod_{i=1}^{r}(\beta_i + 1) \), we get the relation

\[ d(mn) \leq d(m)d(n) \] (4)

with equality only for \( (n, m) = 1 \).

Let now \( m = k, \; n = (k - 1)! \) for \( k \geq 2 \). Then relation (3) gives

\[ d(k!) > d((k - 1)!) \] for all \( k \geq 2 \),

thus proving the assertion that equation (1) always has at least a solution (for \( k = 1 \) one can take \( x = 1 \)).

With the same substitutions, relation (4) yields

\[ d(k!) \leq d((k - 1)!)d(k) \] for \( k \geq 2 \) (6)
Let \( k = p \) (prime) in (6). Since \(((p - 1)!), p) = 1\), we have equality in (6):

\[
\frac{d(p!)}{d((p-1)!)} = 2, \quad p \text{ prime.}
\]  

(7)

3. Since \( S(k!)/k! \to 0 \), \( \frac{S(k!)}{S((k-1)!)} = \frac{k}{k-1} \to 1 \) as \( k \to \infty \), one may ask the similar problems for such limits for other arithmetical functions.

It is well known that

\[
\frac{\sigma(n!)}{n!} \to \infty \text{ as } n \to \infty.
\]  

(8)

In fact, this follows from \( \sigma(k) = \sum_{d|k} d = \sum \frac{k}{d} \), so

\[
\frac{\sigma(n!)}{n!} = \sum_{d|n!} \frac{1}{d} \geq 1 + \frac{1}{2} + \ldots + \frac{1}{n} > \log n,
\]

as it is known.

From the known inequality (\([1]\)) \( \varphi(n)\sigma(n) \leq n^2 \) it follows

\[
\frac{n}{\varphi(n)} > \frac{\sigma(n)}{n},
\]

so \( \frac{n!}{\varphi(n!)} \to \infty \), implying

\[
\frac{\varphi(n!)}{n!} \to 0 \text{ as } n \to \infty.
\]  

(9)

Since \( \varphi(n) > d(n) \) for \( n > 30 \) (see \([2]\)), we have \( \varphi(n!) > d(n!) \) for \( n! > 30 \) (i.e. \( n \geq 5 \)), so, by (9)

\[
\frac{d(n!)}{n!} \to 0 \text{ as } n \to \infty.
\]  

(10)

In fact, much stronger relation is true, since \( \frac{d(n)}{n^\varepsilon} \to 0 \) for each \( \varepsilon > 0 \) (\( n \to \infty \)) (see \([1]\)). From \( \frac{d(n!)}{n!} < \frac{\varphi(n!)}{n!} \) and the above remark on \( \sigma(n!) > n! \log n \), it follows that

\[
\limsup_{n \to \infty} \frac{d(n!)}{n!} \log n \leq 1.
\]  

(11)
These relations are obtained by very elementary arguments. From the inequality
\[ \varphi(n)(\omega(n) + 1) \geq n \] (see [2]) we get
\[ \omega(n!) \to \infty \text{ as } n \to \infty \tag{12} \]
and, since \( \Omega(s) \geq \omega(s) \), we have
\[ \Omega(n!) \to \infty \text{ as } n \to \infty. \tag{13} \]

From the inequality \( nd(n) \geq \varphi(n) + \sigma(n) \) (see [2]), and (8), (9) we have
\[ d(n!) \to \infty \text{ as } n \to \infty. \tag{14} \]

This follows also from the known inequality \( \varphi(n)d(n) \geq n \) and (9), by replacing \( n \) with \( n! \). From \( \sigma(mn) \geq m\sigma(n) \) (see [3]) with \( n = (k - 1)! \), \( m = k \) we get
\[ \frac{\sigma(k!)}{\sigma((k - 1)!)} \geq k \quad (k \geq 2) \tag{15} \]
and, since \( \sigma(mn) \leq \sigma(m)\sigma(n) \), by the same argument
\[ \frac{\sigma(k!)}{\sigma((k - 1)!)} \leq \sigma(k) \quad (k \geq 2). \tag{16} \]

Clearly, relation (15) implies
\[ \lim_{k \to \infty} \frac{\sigma(k!)}{\sigma((k - 1)!)} = +\infty. \tag{17} \]

From \( \varphi(m)\varphi(n) \leq \varphi(mn) \leq m\varphi(n) \), we get, by the above remarks, that
\[ \varphi(k) \leq \frac{\varphi(k!)}{\varphi((k - 1)!)} \leq k, \quad (k \geq 2) \tag{18} \]
implying, by \( \varphi(k) \to \infty \) as \( k \to \infty \) (e.g. from \( \varphi(k) > \sqrt{k} \) for \( k > 6 \)) that
\[ \lim_{k \to \infty} \frac{\varphi(k!)}{\varphi((k - 1)!)^2} = +\infty. \tag{19} \]
By writing \( \sigma(k!) - \sigma((k - 1)!) = \sigma((k - 1)!) \left[ \frac{\sigma(k!)}{\sigma((k - 1)!) - 1} \right] \), from (17) and \( \sigma((k - 1)!) \to \infty \) as \( k \to \infty \), we trivially have:

\[
\lim_{k \to \infty} [\sigma(k!) - \sigma((k - 1)!)] = +\infty.
\] (20)

In completely analogous way, we can write:

\[
\lim_{k \to \infty} [\varphi(k!) - \varphi((k - 1)!)] = +\infty.
\] (21)

4. Let us remark that for \( k = p \) (prime), clearly \( ((k - 1)!, k) = 1 \), while for \( k = \text{composite} \), all prime factors of \( k \) are also prime factors of \((k - 1)!\). Thus

\[
\omega(k!) = \begin{cases} 
\omega((k - 1)!k) = \omega((k - 1)!)) + \omega(k) & \text{if } k \text{ is prime} \\
\omega((k - 1)!) & \text{if } k \text{ is composite } (k \geq 2).
\end{cases}
\]

Thus

\[
\omega(k!) - \omega((k - 1)!) = \begin{cases} 
1, & \text{for } k \text{ is prime} \\
0, & \text{for } k \text{ is composite}
\end{cases}
\] (22)

Thus we have

\[
\limsup_{k \to \infty} [\omega(k!) - \omega((k - 1)!)] = 1
\]

\[
\liminf_{k \to \infty} [\omega(k!) - \omega((k - 1)!)] = 0
\] (23)

Let \( p_n \) be the nth prime number. From (22) we get

\[
\frac{\omega(k!)}{\omega((k - 1)!)} - 1 = \begin{cases} 
\frac{1}{n - 1}, & \text{if } k = p_n \\
0, & \text{if } k \text{ is composite.}
\end{cases}
\]

Thus, we get

\[
\lim_{k \to \infty} \frac{\omega(k!)}{\omega((k - 1)!)} = 1.
\] (24)

The function \( \Omega \) is totally additive, so

\[
\Omega(k!) = \Omega((k - 1)!k) = \Omega((k - 1)!)) + \Omega(k),
\]
giving

\[ \Omega(k!) - \Omega((k-1)!) = \Omega(k). \]  

(25)

This implies

\[ \limsup_{k \to \infty} [\Omega(k!) - \Omega((k-1)!)] = +\infty \]  

(26)

(take e.g. \( k = 2^m \) and let \( m \to \infty \), and

\[ \liminf_{k \to \infty} [\Omega(k!) - \Omega((k-1)!) = 2 \]

(take \( k = \text{prime} \)).

For \( \Omega(k!)/\Omega((k-1)!)) \) we must evaluate

\[ \frac{\Omega(k)}{\Omega((k-1)!)} = \frac{\Omega(k)}{\Omega(1) + \Omega(2) + \ldots + \Omega(k-1)}. \]

Since \( \Omega(k) \leq \frac{\log k}{\log 2} \) and by the theorem of Hardy and Ramanujan (see \([1]\)) we have

\[ \sum_{n \leq x} \Omega(n) \sim x \log \log x \quad (x \to \infty) \]

so, since \( \frac{\log k}{(k-1) \log \log(k-1)} \to 0 \) as \( k \to \infty \), we obtain

\[ \lim_{k \to \infty} \frac{\Omega(k!)}{\Omega((k-1)!)} = 1. \]

(27)

5. Inequality (18) applied for \( k = p \) (prime) implies

\[ \lim_{p \to \infty} \frac{1}{p} \cdot \frac{\varphi(p!)}{\varphi((p-1)!) = 1}. \]

(28)

This follows by \( \varphi(p) = p - 1 \). On the other hand, let \( k > 4 \) be composite. Then, it is known (see \([1]\)) that \( k|(k-1)! \). So \( \varphi(k!) = \varphi((k-1)!k) = k\varphi((k-1)!) \), since \( \varphi(mn) = m\varphi(n) \) if \( m|n \). In view of (28), we can write

\[ \lim_{k \to \infty} \frac{1}{k} \cdot \frac{\varphi(k!)}{\varphi((k-1)!)} = 1. \]

(29)
For the function $\sigma$, by (15) and (16), we have for $k = p$ (prime) that $p \leq \frac{\sigma(p!)}{\sigma((p - 1)!) \leq \sigma(p) = p + 1$, yielding

$$\lim_{p \to \infty} \frac{1}{p} \cdot \frac{\sigma(p!)}{\sigma((p - 1)!) = 1.}$$

(30)

In fact, in view of (15) this implies that

$$\liminf_{k \to \infty} \frac{1}{k} \cdot \frac{\sigma(k!)}{\sigma((k - 1)!) = 1.}$$

(31)

By (6) and (7) we easily obtain

$$\limsup_{k \to \infty} \frac{d(k!)}{d(k)d((k - 1)!)} = 1.$$  

(32)

In fact, inequality (6) can be improved, if we remark that for $k = p$ (prime) we have $d(k!) = d((k - 1)!)$, while for $k = \text{composite, } k > 4$, it is known that $k|(k - 1)!$. We apply the following

**Lemma.** If $n|m$, then

$$\frac{d(mn)}{d(m)} \leq \frac{d(n^2)}{d(n)}.$$  

(33)

**Proof.** Let $m = \prod p^\alpha \prod q^\beta$, $n = \prod p^{\alpha'}$ ($\alpha' \leq \alpha$) be the prime factorizations of $m$ and $n$, where $n|m$. Then

$$\frac{d(mn)}{d(m)} = \frac{\prod (\alpha + \alpha' + 1) \prod (\beta + 1)}{\prod (\alpha + 1) \prod (\beta + 1)} = \prod \left( \frac{\alpha + \alpha' + 1}{\alpha + 1} \right).$$

Now $\frac{\alpha + \alpha' + 1}{\alpha + 1} \leq \frac{2\alpha' + 1}{\alpha' + 1}$ $\Rightarrow$ $\alpha' \leq \alpha$ as an easy calculations verifies. This immediately implies relation (33).

By selecting now $n = k$, $m = (k - 1)!$, $k > 4$ composite we can deduce from (33):

$$\frac{d(k!)}{d((k - 1)!) \leq \frac{d(k^2)}{d(k)}.}$$

(34)

By (4) we can write $d(k^2) < (d(k))^2$, so (34) represents indeed, a refinement of relation (6).
References

