SOLUTION OF TWO QUESTIONS CONCERNING
THE DIVISOR FUNCTION AND THE PSEUDO-SMARANDACHE FUNCTION

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Abstract In this paper we completely solve two questions concerning the divisor function and the pseudo-Smarandache function.

Key words divisor function, pseudo-Smarandache function, functional equation

1 Introduction

Let \( \mathbb{N} \) be the set of all positive integers. For any \( n \in \mathbb{N} \), let
\[
(1) \quad d(n) = \sum_{d|n} 1,
\]
\[
(2) \quad Z(n) = \min \{ a \mid a \in \mathbb{N}, n \mid \sum_{j=1}^{a} j \}.
\]
Then \( d(n) \) and \( Z(n) \) are called the divisor function and the pseudo-Smarandache function of \( n \), respectively. In [1], Ashbacher posed the following unsolved questions.

**Question 1** How many solutions \( n \) are there to the functional equation.
\[
(3) \quad Z(n) = d(n), n \in \mathbb{N}?
\]

**Question 2** How many solutions \( n \) are there to the functional equation.
In this paper we completely solve the above questions as follows.

**Theorem 1** The equation (3) has only the solutions \( n = 1, 3 \) and 10.

**Theorem 2** The equation (4) has only the solution \( n = 56 \).

### 2 Proof of Theorem 1

A computer search showed that (3) has only the solutions \( n = 1, 3 \) and 10 with \( n \leq 10000 \) (see [1]).

We now let \( n \) be a solution of (3) with \( n \neq 1, 3 \) or 10. Then we have \( n > 10000 \). Let

\[
 n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}
\]

be the factorization of \( n \). By [2, Theorem 273], we get from (1) and (5) that

\[
d(n) = (r_1 + 1)(r_2 + 1) \cdots (r_k + 1).
\]

On the other hand, since \( \sum_{j=1}^{\infty} j = a (a + 1)/2 \) for any \( a \in \mathbb{N} \), we see from (2) that \( n | Z(n)(Z(n) + 1)/2 \). It implies that \( Z(n)(Z(n) + 1)/2 \geq n \). So we have

\[
 Z(n) \geq \sqrt{2n + \frac{1}{4} - \frac{1}{2}}
\]

Hence, by (3), (5), (6) and (7), we get

\[
 1 \geq \sqrt{2} \prod_{i=1}^{\infty} \frac{p_i^{r_i/2}}{r_i + 1} - \frac{1}{2} \prod_{i=1}^{\infty} \frac{1}{r_i + 1}
\]

If \( p_1 > 3 \), then from (8) we get \( p_1 \geq 5 \) and

\[
 1 \geq \sqrt{2} \left( \frac{\sqrt{5}}{2} \right)^k - \frac{1}{2^{k+1}} > 1,
\]
a contradiction. Therefore, if (8) holds, then either \( p_1 = 2 \) or \( p_1 = 3 \). By the same method, then \( n \) must satisfy one of the following conditions.

(i) \( p_1 = 2 \) and \( r_1 \leq 4 \).

(ii) \( p_1 = 3 \) and \( r_1 = 1 \).

However, by (8), we can calculate that \( n < 10000 \), a contradiction. Thus, the theorem is proved.

3 Proof of Theorem 2

A computer search showed that (4) has only the solution \( n = 56 \) with \( n \leq 10000 \) (see \[^{11}\]). We now let \( n \) be a solution of (4) with \( n \neq 56 \). Then we have \( n > 10000 \). We see from (4) that

\[(9) \quad Z(n) \equiv -d(n) \pmod{n}\]

It implies that

\[(10) \quad Z(n) + 1 \equiv 1 - d(n) \pmod{n}\]

By the proof of Theorem 1, we have \( n \mid Z(n)(Z(n) + 1)/2 \), by (2). It can be written as

\[(11) \quad Z(n)(Z(n) + 1) \equiv 0 \pmod{n} .\]

Substituting (9) and (10) into (11), we get

\[(12) \quad d(n)(d(n) - 1) \equiv 0 \pmod{n} .\]

Notice that \( d(n) > 1 \) if \( n > 1 \). We see from (12) that

\[(13) \quad (d(n))^2 > n \]

Let (5) be the factorization of \( n \). By (5), (6) and (13), we obtain

\[(14) \quad 1 > \prod_{i=1}^{r} \frac{p_i^{r_i}}{(r_i + 1)^2} \]
On the other hand, it is a well known fact that \( Z(p^r) \geq p^r - 1 > (r + 1)^2 \) for any prime power \( p^r \) with \( p^r > 32 \). We find from (14) that \( k \geq 2 \).

If \( p_1 > 3 \), then \( p_i/(r_i + 1)^2 \geq 5/4 > 1 \) for \( i = 1, 2, \ldots, k \). It implies that if (14) holds, then either \( p_1 = 2 \) or \( p_1 = 3 \). By the same method, then \( n \) must satisfy one of the following conditions:

(i) \( p_1 = 2, p_2 = 3 \) and \((r_1, r_2) = (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (1, 2), (2, 2), (3, 2), (4, 2) \) or \((5, 2)\).

(ii) \( p_1 = 2, p_2 > 3 \) and \( r_1 \leq 5 \).

(iii) \( p_1 = 3 \) and \( r_1 = 1 \).

However, by (14), we can calculate that \( n < 10000 \), a contradiction. Thus, the theorem is proved.

References


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