ON SOME SERIES INVOLVING
SMARANDACHE FUNCTION

by
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The study of infinite series involving Smarandache function is one of the most interesting aspects of analysis.

In this brief article we give only a bare introduction to it.

First we prove that the series \( \sum_{k=2}^{\infty} \frac{S(k)}{(kH)!} \) converges and has the sum \( e^{-\frac{5}{2}} \). Let us denote
\[
E_n = \sum_{k=2}^{n} \frac{S(k)}{(k+1)!} < 1
\]

by \( E_n \). We show that
\[
E_{n+1} - \frac{5}{2} < \sum_{k=2}^{n} \frac{S(k)}{(k+1)!} < \frac{1}{2}
\]
as follows:

\[
\sum_{k=2}^{n} \frac{k}{(k+1)!} = \sum_{k=2}^{n} \left( \frac{1}{k!} - \frac{1}{(k+1)!} \right) = \sum_{k=2}^{n} \frac{1}{k!} \frac{1}{(k+1)!} = \frac{1}{2!} - \frac{1}{(n+1)!}
\]

\( S(k) < k \) implies that
\[
\sum_{k=2}^{n} \frac{S(k)}{(k+1)!} \leq \sum_{k=2}^{n} \frac{k}{(k+1)!} = \frac{1}{2} - \frac{1}{(n+1)!} < \frac{1}{2}
\]
On the other hand \( k \geq 2 \) implies that \( S(k) > \frac{1}{k} \) and consequently:

\[
\sum_{k=2}^{n} \frac{S(k)}{(k+1)!} > \sum_{k=2}^{n} \frac{1}{(k+1)!} = \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{n+1}! = E_{n+1} - \frac{5}{2}.
\]

It follows that \( E_{n+1} - \frac{5}{2} < \sum_{k=2}^{n} \frac{S(k)}{(k+1)!} < \frac{1}{2} \) and therefore

\[
\sum_{k=2}^{\infty} \frac{S(k)}{(k+1)!}
\]

is a convergent series with sum \( \sigma \epsilon \left[ e^{-\frac{5}{2}}, \frac{1}{2} \right] \).

REMARK: Some of inequalities \( S(k) \leq k \) are strictly and \( k \geq S(k)+1 \), \( S(k) \geq 2 \). Hence \( \sigma \epsilon \left[ e^{-\frac{5}{2}}, \frac{1}{2} \right] \).

We can also check that \( \sum_{k=2}^{n} \frac{S(k)}{(k-r)!} \), \( r \in \mathbb{N} \) and \( \sum_{k=2}^{\infty} \frac{S(k)}{(k+r)!} \), \( r \in \mathbb{N} \),

are both convergent as follows:

\[
\sum_{k=2}^{n} \frac{S(k)}{(k-r)!} \leq \sum_{k=2}^{n} \frac{k}{(k-r)!} = \frac{r}{0!} + \frac{r+1}{1!} + \frac{r+2}{2!} + \ldots + \frac{r+(n-r)}{(n-r)!} =
\]

\[
=r \left( \frac{1}{0!} + \frac{1}{1!} + \ldots + \frac{1}{(n-r)!} \right) + \left( \frac{1}{1!} + \frac{2}{2!} + \ldots + \frac{n-r}{(n-r)!} \right) = rE_{n-r} + E_{n-r-1}
\]

We get \( \sum_{k=2}^{n} \frac{S(k)}{(k-r)!} < rE_{n-r} + E_{n-r-1} \), which that \( \sum_{k=2}^{\infty} \frac{S(k)}{(k-r)!} \)

converges.

Also we have \( \sum_{k=2}^{\infty} \frac{S(k)}{(k+r)!} < \infty \), \( r \in \mathbb{N} \).

Let us define the set \( M_2 = \{ m \in \mathbb{N} : m = \frac{n}{2}, n \in \mathbb{N}, n \geq 3 \} \).

If \( m \in M_2 \) it is obvious that \( S(m) = n \), \( m = \frac{n}{2} \), \( m \in M_2 \rightarrow \frac{m}{S(m)!} = \frac{n!}{2} \).

So, \( \sum_{m=1}^{\infty} \frac{m}{S(m)!} = \infty \) and therefore \( \sum_{k=2}^{\infty} \frac{k}{S(k)!} = \infty \).

A problem: test the convergence behaviour of the series \( \sum_{k=2}^{\infty} \frac{1}{S(k)!} \).
REFERENCES


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