The Smarandache Function is defined as $S(n) = k$. Where $k$ is the smallest integer such that $n$ divides $k!$

Let us define $S_c(n)$ **Smarandache Reciprocal Function** as follows:

$S_c(n) = x$ where $x + 1$ does not divide $n!$ and for every $y \leq x$, $y \mid n!$

**THEOREM-I.**

If $S_c(n) = x$, and $n \leq 3$, then $x + 1$ is the smallest prime greater than $n$.

**PROOF:** It is obvious that $n!$ is divisible by $1, 2, \ldots$ up to $n$. We have to prove that $n!$ is also divisible by $n + 1, n + 2, \ldots n + m (= x)$, where $n + m + 1$ is the smallest prime greater than $n$. Let $r$ be any of these composite numbers. Then $r$ must be factorable into two factors each of which is $\geq 2$. Let $r = p.q$, where $p, q \geq 2$. If one of the factors (say $q$) is $\geq n$ then $r = p.q \geq 2n$. But according to the Bertrand's postulate there must be a prime between $n$ and $2n$, there is a contradiction here since all the numbers from $n + 1$ to $n + m$ ($n + 1 \leq r < n + m$) are assumed to be composite. Hence neither of the two factors $p, q$ can be $\geq n$. So each must be $< n$. Now there are two possibilities:
Case-I $p \neq q$.

In this case as each is $< n$ so \( p \cdot q = r \) divides $n!$

Case-II $p = q = \text{prime}$

In this case $r = p^2$ where $p$ is a prime. There are again three possibilities:

(a) $p \geq 5$. Then $r = p^2 > 4p \Rightarrow 4p < r < 2n \Rightarrow 2p < n$. Therefore both $p$ and $2p$ are less than $n$ so $p^2$ divides $n!$

(b) $p = 3$, Then $r = p^2 = 9$ Therefore $n$ must be 7 or 8. and 9 divides 7! and 8!.

(c) $p = 2$, then $r = p^2 = 4$. Therefore $n$ must be 3. But 4 does not divide 3!, Hence the theorem holds for all integral values of $n$ except $n = 3$. This completes the proof.

Remarks: Readers can note that $n!$ is divisible by all the composite numbers between $n$ and $2n$.

Note: We have the well known inequality $S(n) \leq n$. \(---------(2)\)

From the above theorem one can deduce the following inequality.

If $p_r$ is the $r^{\text{th}}$ prime and $p_r \leq n < p_{r+1}$ then $S(n) \leq p_r$. (A slight improvement on (2)).
i.e. $S(p_r) = p_r$, $S(p_r + 1) < p_r$, $S(p_r + 2) < p_r$, \ldots $S(p_{r+1} - 1) < p_r$, $S(p_{r+1}) = p_{r+1}$

Summing up for all the numbers $p_r \leq n < p_{r+1}$ one gets

$$\sum_{t=0}^{p_{r+1} - p_r - 1} S(p_r + t) \leq (p_{r+1} - p_r) p_r$$

Summing up for all the numbers up to the $s^{th}$ prime, defining $p_0 = 1$, we get

$$\sum_{k=1}^{p_s} S(k) \leq \sum_{r=0}^{s} (p_{r+1} - p_r) p_r \quad \text{-------(3)}$$

More generally from Ref. [1] following inequality on the $n$th partial sum of the Smarandache (Inferior) Prime Part Sequence directly follows.

**Smarandache (Inferior) Prime Part Sequence**

For any positive real number $n$ one defines $p_p(n)$ as the largest prime number less than or equal to $n$. In [1] Prof. Krassimir T. Atanassov proves that the value of the $n^{th}$ partial sum of this sequence $X_n = \sum_{k=1}^{n} p_p(k)$ is given by

$$X_n = \sum_{k=2}^{\pi(n)} (p_k - p_{k-1}) p_{k-1} + (n - p_{\pi(n)} + 1) p_{\pi(n)} \quad \text{-------(4)}$$

From (3) and (4) we get
\[
\sum_{k=1}^{n} S(k) \leq X_n
\]

REFERENCES:


[2] "The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texas at Austin, USA.