Some Considerations Concerning the Sumatory Function Associated to Generalised Smarandache Function

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Let us denote by $V$ the least common multiple, by $\Lambda$ the greatest common divisor and $\Lambda = \min, V = \max$. It is known that $N_0 = (N^*, \Lambda, V)$ and $N_d = \left( N^*, \Lambda_d, V_d \right)$ are lattices. The order on the set $N^*$ is $n_1 \leq n_2 \iff n_1 \Lambda n_2 = n_1$. Corresponding to the first of these lattices and it is known that this is a total order. But the order $\leq_d$ induced on the same set by $\Lambda_d$ and $V_d$ is only a partial order.

Let $\sigma : N_0 \rightarrow N_d$ (1) a sequence of positive integers defined on the set $N^*$. The sequence (1) is said to be a multiplicatively convergent to zero sequence (mcz) if:

$$\forall n \in N^*, \exists m_1 \in N^*, \forall m > m_1 \Rightarrow n \leq \sigma(m)$$

(2).

The sequence

$$\sigma : N_d \rightarrow N_d$$

(3)

is said to be a divisibility sequence (ds) if: $n \leq m \Rightarrow \sigma(n) \leq \sigma(m)$ and it is said to be a strong divisibility sequence (sds) if:

$$\sigma(n \wedge m) = \sigma(n) \wedge \sigma(m) \text{ for every } n, m \in N^*$$

(4).

Let the lattices $N_0$ and $N_d$. We'll use the following notations:

(a) a sequence $\sigma_{oo} : N_0 \rightarrow N_0$ is a (oo) - sequences;

(b) a sequence $\sigma_{od} : N_0 \rightarrow N_d$ is a (od) - sequences;

(c) a sequence $\sigma_{do} : N_d \rightarrow N_0$ is a (do) - sequences;

(d) a sequence $\sigma_{dd} : N_d \rightarrow N_d$ is a (dd) - sequences.

Then $\Lambda(\text{do})$ - sequence $\sigma_{do}$ the monotonicity yields:

$$(m_{\sigma_{do}}) \forall n_1, n_2 \in N^*, n_1 \leq n_2 \Rightarrow \sigma_{do}(n_1) \leq \sigma_{do}(n_2)$$

(5)

and the condition of convergence to infinity is:

$$(c_{\sigma_{do}}) \forall n \in N^*, \exists m_{\sigma_{do}} \in N^*, \forall m \leq m_{\sigma_{do}} \Rightarrow \sigma_{do}(m) \geq n$$

(6).

Analogously, for a (dd) - sequence $\sigma_{dd}$ the monotonicity yields:

$$(m_{\sigma_{dd}}) \forall n_1, n_2 \in N^*, n_1 \leq n_2 \Rightarrow \sigma_{dd}(n_1) \leq \sigma_{dd}(n_2)$$

(7)

and the convergence to zero is:

$$(c_{\sigma_{dd}}) \forall n \in N^*, \exists m_{\sigma_{dd}} \in N^*, \forall m \geq m_{\sigma_{dd}} \Rightarrow \sigma_{dd}(m) \leq n$$

(8).

To each sequence $\sigma_{ij}$, with $i, j \in \{0, d\}$, satisfying the condition $(c_{ij})$, one may attach a sequence $S_{ij}$ (a generalised Smarandache function) defined by:

$$S_{ij} = \min \left\{ m_{\sigma_{ij}} : m_{\sigma_{ij}} \text{ is given by the condition } (c_{ij}) \right\}$$

(9).

For the properties the functions $S_{ij}$, see [2].

It is said that for every numerical function $f$ it can be attached the sumatory function:

$$F(x) = \sum_{d \mid x} \lambda(d)$$

(10)

The function $\lambda$ is expressed as:

$$\lambda(n) = \sum_{d \mid n} \mu(d) F(\frac{n}{d})$$

(11)

where $\mu$ is the Mobius function ($\mu(1) = 1, \mu(n) = 0$ if $n$ is divisible by the square of a prime number, $\mu(n) = (-1)^k$ if $n$ the product of $k$ different prime numbers).

If $f$ is the a generalised Smarandache function, $S_{ij}$ then
\[ F'_\phi(n) = \sum_{d|n} S_\phi(d), \text{ } i, j \in \{0, d\}. \]

Now let us consider \( n = p_1 p_2 \ldots p_k \), with \( p_1 < p_2 < \ldots < p_k \) primes number and \( S_\phi(p_1) \leq S_\phi(p_2) \leq \ldots \leq S_\phi(p_k) \), for example. If \( i=0, j=d \), then \( S_\phi(n_1 d n_2) = S_\phi(n_1) \vee S_\phi(n_2) \) and

\[ F_\phi^S(n) = S_\phi(1) + \sum_{b=1}^{k} S_\phi(p_b) + \sum_{b=1}^{k} S_\phi(p_b p_2 p_3) + \sum_{b=1}^{k} S_\phi(p_b p_2 p_3 p_4) + \ldots S_\phi(n). \]

It result:

\[ F_\phi^S(1) = S_\phi(1); \]

\[ F_\phi^S(p_1) = S_\phi(1) + S_\phi(p_1) = F_\phi^S(1) + 2^0 S_\phi(p_1); \]

\[ F_\phi^S(p_1 p_2) = S_\phi(1) + S_\phi(p_1) + S_\phi(p_2) + S_\phi(p_1 p_2) = S_\phi(1) + 2 S_\phi(p_1) + 2 S_\phi(p_2) = F_\phi^S(p_1) + 2 S_\phi(p_2); \]

\[ F_\phi^S(p_1 p_2 p_3) = F_\phi^S(p_1 p_2) + S_\phi(p_3); \]

\[ F_\phi^S(p_1 p_2 p_3 p_4) = F_\phi^S(p_1 p_2 p_3) + 2 S_\phi(p_4); \]

\[ \ldots \]

That is \( F_\phi^S(p_1 p_2 \ldots p_k) = S_\phi(1) + \frac{k}{2} 2^{i-1} S_\phi(p_i). \)

The equality (11) becomes:

\[ S_\phi(n) = \sum_{d|n} \mu(d) F_\phi^S(d) = \]

\[ = F_\phi^S(1) - \sum_i F_\phi^S(d_i) + \sum_j F_\phi^S(d_j) + \ldots \]

with \( F_\phi^S(d_i) = F_\phi^S(p_1 p_2 \ldots p_i) = \frac{k}{2} 2^{i-1} S_\phi(p_i) + \sum_{p_{i+1}}^{k} 2^{i-1} S_\phi(p_i) = \]

\[ = F_\phi^S(p_1 p_2 \ldots p_i) + 2^{i-1} F_\phi^S(p_{i+1} \ldots p_k). \]

Analogously,

\[ F_\phi^S(p_i) = 2^{i-1} F_\phi^S(p_{i+1} \ldots p_k) + 2^{i-1} F_\phi^S(p_i p_k) \]

\[ = \sum_{b=1}^{k} S_\phi(p_i) + \sum_{b=1}^{k} 2^{k-b} S_\phi(p_b) + \frac{k}{2} 2^{k-1} S_\phi(p_i). \]

In particularity, for \( n = p^r \), \( p \) prime number, it result:

\[ S_\phi(n^r) = \sum_{d|n^r} \mu(p^r) F_\phi^S(p^r) = F_\phi^S(p^r) - F_\phi^S(p^{r-1}). \]

If \( n = p^r q^s \) with max \( \{ S_\phi(p), \ldots, S_\phi(p^r) \} \leq \min \{ S_\phi(q), \ldots, S_\phi(q^s) \} \), then

\[ F_\phi^S(p^r q^s) = F_\phi^S(p^r) + (a+1) F_\phi^S(q^s). \]

If \( i=d, \ j=d \) and if \( \sigma_{ae} \) is a (sds) satisfying the condition \( (c_{ae}) \), then

\[ S_{ae}(n_1 d n_2) = S_{ae}(n_1) \vee S_{ae}(n_2) + \]

\[ + \sum_{b=1}^{k} S_{ae}(p_b) + \sum_{b=1}^{k} \left[ S_{ae}(p_b) \vee S_{ae}(p_{b+t}) \right] + \ldots S_{ae}(n) \]

\[ + \sum_{b=1}^{k} S_{ae}(p_b) \vee S_{ae}(p_{b+t}) \] (13)

and \( F_{ae}(n) = S_{ae}(1) + \frac{k}{2} S_{ae}(p_b) + \frac{k}{2} \left[ S_{ae}(p_b) \vee S_{ae}(p_{b+t}) \right] + \ldots S_{ae}(n) \)

\[ S_{ae}(p^r) + F_{ae}(p^r) - F_{ae}(p^{r-1}). \]

Example: The Fibonacci sequence \( (F_n)_{n=0}^\infty \) defined by \( F_{n+1} = F_n + F_{n-1} \), with \( F_1 = F_2 = 1 \) is a (sds), so for the generalised Smarandache function \( S_F \) attached to this sequence we have:

\[ S_F(n_1 d n_2) = S_F(n_1) \vee S_F(n_2), \]

and the calculus of \( S_F(n) \) is reduced to the calculus of \( S_F(p^r) \), with \( p \) a prime number. For instance:

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$F_{dd}^S(4) = 10, F_{dd}^S(8) = 16, F_{dd}^S(16) = 28, F_{dd}^S(15) = 30.$

References:


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