THE EQUATION $S(1.2)+S(2.3)+\cdots+S(n(n+1))=S(n(n+1)(n+2)/3)$

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Abstract. For any positive integer $a$, let $S(a)$ be the Smarandache function of $a$. In this paper we prove that the title equation has only the solution $n=1$.

Key words: Smarandache function, diophantine equation

Let $N$ be the set of all positive integers. For any positive integer $a$, let $S(a)$ be the Smarandache function of $a$. Recently, Bencze [1] proposed the following problem:

Problem Solve the equation

(1) $S(1\cdot2)+S(2\cdot3)+\cdots+S(n(n+1))=S\left(\frac{1}{3}n(n+1)(n+2)\right), n\in N.$

In this paper we completely solve the above-mentioned problem as follows.

Theorem The equation (1) has only the solution $n=1$.

Proof By the definition of the Smarandache function (see [2]), we have $S(1)=1$, $S(2)=2$ and

(2) $S(a)\geq 3, a\geq 3.$

Since $S(1.2)=S(1.2.3/3)=S(2)$, the equation (1) has a solution $n=1$.

Let $n$ be a solution of (1) with $n>1$. Then, by (2), we get

(3) $S(1\cdot2)+S(2\cdot3)+\cdots+S(n(n+1))\geq 2+3(n-1)=3n-1.$

Therefore, by (1) and (3), we obtain
(4) \[ S\left(\frac{1}{3}n(n+1)(n+2)\right) \geq 3n-1. \]

On the other hand, since \((n+2)! = 1.2\cdots\cdot n(n+1)(n+2)\), we get

(5) \[ \frac{1}{3}n(n+1)(n+2) \mid (n+2)! . \]

We see from (5) that

(6) \[ S\left(\frac{1}{3}n(n+1)(n+2)\right) \leq n + 2. \]

The combination of (4) and (6) yields

(7) \[ n + 2 \geq 3n - 1, \]

whence we get \(n \leq 3/2 < 2\). Since \(n \geq 2\), it is impossible. Thus, (1) has no solutions \(n\) with \(n > 1\). The theorem is proved.

References


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