THE THIRD AND FOURTH CONSTANTS OF SMARANDACHE

by

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In the present note we prove the divergence of some series involving the Smarandache function, using an unitary method, and then we prove that the series

\[ \sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\ldots S(n)} \]

is convergent to a number \( s \in (71/100, 101/100) \) and we study some applications of this series in the Number Theory (third constant of Smarandache).

The Smarandache Function \( S : \mathbb{N}^* \rightarrow \mathbb{N} \) is defined \([1]\) such that \( S(n) \) is the smallest integer \( k \) with the property that \( k! \) is divisible by \( n \).

Proposition 1. If \( (x_n)_{n \geq 1} \) is a strict increasing sequence of natural numbers, then the series:

\[ \sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}, \]

where \( S \) is the Smarandache function, is divergent.

Proof. We consider the function \( f : [x_n, x_{n+1}] \rightarrow \mathbb{R} \), defined by \( f(x) = \ln \ln x \). It fulfils the conditions of the Lagrange's theorem of finite increases. Therefore there is \( c_n \in (x_n, x_{n+1}) \) such that:

\[ \ln \ln x_{n+1} - \ln \ln x_n = \frac{1}{c_n \ln c_n} (x_{n+1} - x_n). \]  

Because \( x_n < c_n < x_{n+1} \), we have:

\[ \frac{x_{n+1} - x_n}{x_{n+1} \ln x_{n+1}} < \ln \ln x_{n+1} - \ln \ln x_n < \frac{x_{n+1} - x_n}{x_n \ln x_n}, \quad (\forall) n \in \mathbb{N}, \]

if \( x_n = 1 \).
We know that for each \( n \in \mathbb{N}^* \setminus \{1\} \), \( \frac{S(n)}{n} \leq 1 \), i.e.

\[
0 < \frac{S(n)}{n \ln n} \leq \frac{1}{\ln n}, \tag{4}
\]

from where it results that \( \lim_{n \to \infty} \frac{S(n)}{n \ln n} = 0 \). Hence there exists \( k > 0 \) such that

\[
\frac{S(n)}{n \ln n} < k, \quad \text{i.e., } \quad n \ln n > \frac{S(n)}{k} \quad \text{for any } n \in \mathbb{N}^*,
\]

so

\[
\frac{1}{x_n \ln x_n} < \frac{k}{S(x_n)}. \tag{5}
\]

Introducing (5) in (3) we obtain:

\[
\ln \ln x_{n+1} - \ln \ln x_n < k \frac{x_{n+1} - x_n}{S(x_n)}, \quad (\forall)n \in \mathbb{N}^* \setminus \{1\}. \tag{6}
\]

Summing up after \( n \) it results:

\[
\sum_{m=1}^{n} \frac{x_{m+1} - x_m}{S(x_m)} > \frac{1}{k} \ln \ln x_{n+1} - \ln \ln x_1.
\]

Because \( \lim_{m \to \infty} x_m = \infty \) we have \( \lim_{m \to \infty} \ln \ln x_m = \infty \), i.e., the series:

\[
\sum_{m=1}^{\infty} \frac{x_{m+1} - x_m}{S(x_m)}
\]

is divergent. The Proposition 1 is proved.

**Proposition 2.** Series \( \sum_{n=2}^{\infty} \frac{1}{S(n)} \), where \( S \) is the Smarandache function, is divergent.

**Proof.** We use Proposition 1 for \( x_n = n \).

**Remarks.** 1) If \( x_n \) is the \( n \)-th prime number, then the series \( \sum_{m=1}^{\infty} \frac{x_{m+1} - x_m}{S(x_m)} \) is divergent.

2) If the sequence \( (x_n)_{n \geq 1} \) forms an arithmetical progression of natural numbers, then the series \( \sum_{m=1}^{\infty} \frac{1}{S(x_m)} \) is divergent.

3) The series \( \sum_{m=1}^{\infty} \frac{1}{S(2n+1)} \), \( \sum_{m=1}^{\infty} \frac{1}{S(4n+1)} \) etc., are all divergent.
In conclusion, Proposition 1 offers us an unitary method to prove that the series having one of the precedent aspects are divergent.

**Proposition 3.** The series:

$$\sum_{n=2}^{\infty} \frac{1}{S(2) \cdot S(3) \cdots S(n)}$$

where $S$ is the Smarandache function, is convergent to a number $s \in (71/100, 101/100)$.

**Proof.** From the definition of the Smarandache function it results $S(n) \leq n!$, $(\forall)n \in \mathbb{N}^* \setminus \{1\}$, so $\frac{1}{S(n)} \geq \frac{1}{n!}$.

Summing up, beginning with $n = 2$ we obtain:

$$\sum_{n=2}^{\infty} \frac{1}{S(2) \cdot S(3) \cdots S(n)} \geq \sum_{n=2}^{\infty} \frac{1}{n!} = e - 2.$$

The product $S(2) \cdot S(3) \cdots S(n)$ is greater than the product of prime numbers from the set $\{1, 2, \ldots, n\}$, because $S(p) = p$, for $p$ prime number. Therefore:

$$\frac{1}{\prod_{i=2}^{n} S(i)} < \frac{1}{\prod_{i=1}^{n} p_i}, \quad (7)$$

where $p_k$ is the biggest prime number smaller or equal to $n$.

There are the inequalities:

$$S = \sum_{n=2}^{\infty} \frac{1}{S(2) S(3) \cdots S(n)} = \frac{1}{S(2)} + \frac{1}{S(2) S(3)} + \frac{1}{S(2) S(3) S(4)} + \cdots +$$

$$+ \frac{1}{S(2) S(3) \cdots S(k)} + \cdots < \frac{1}{2} + \frac{2}{2 \cdot 3} + \frac{2}{2 \cdot 3 \cdot 5} + \frac{4}{2 \cdot 3 \cdot 5 \cdot 7} +$$

$$+ \frac{2}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11} + \cdots + \frac{P_{k+1} - P_k}{P_1 P_2 \cdots P_k} + \cdots \quad (8)$$

Using the inequality $p_1 p_2 \cdots p_k > p_{k+1}^3$, $(\forall) k \geq 5 \quad [2]$, we obtain:
\[ S < \frac{1}{2} + \frac{1}{3} + \frac{1}{15} + \frac{2}{105} + \frac{1}{p_6} + \frac{1}{p_7} + \ldots + \frac{1}{p_{k+1}} \quad (9) \]

We note \( P = \frac{1}{p_6} + \frac{1}{p_7} + \ldots \) and observe that \( P < \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{15^2} + \ldots \).

It results:

\[ P < \frac{\pi^2}{6} - \left( 1 - \frac{1}{2^2} + \frac{1}{3^2} + \ldots - \frac{1}{12^2} \right), \]

where

\[ \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \quad \text{(EULER)}. \]

Introducing in (9) we obtain:

\[ S < \frac{1}{2} + \frac{1}{3} + \frac{1}{15} + \frac{2}{105} + \frac{\pi^2}{6} - 1 - \frac{1}{2^2} - \frac{1}{3^2} - \ldots - \frac{1}{12^2}. \]

Estimating with an approximation of an order not more than \( \frac{1}{10^2} \), we find:

\[ 0.71 < \sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\ldots S(n)} < 1.01. \quad (10) \]

The Proposition 3 is proved.

Remark. Giving up at the right increase from the first terms in the inequality (8) we can obtain a better right framing:

\[ \sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\ldots S(n)} < 0.97. \quad (11) \]

Proposition 4. Let \( \alpha \) be a fixed real number, \( \alpha \geq 1 \). Then the series

\[ \sum_{n=2}^{\infty} \frac{n^\alpha}{S(2)S(3)\ldots S(n)} \]

is convergent (fourth constant of Smarandache).

Proof. Be \( (p_k)_{k \geq 1} \) the sequence of prime numbers. We can write:
\[ \frac{2^a}{S(2)} = \frac{2^a}{2} = 2^{a-1} \]
\[ \frac{3^a}{S(2)S(3)} = \frac{3^a}{p_1 p_2} \]
\[ \frac{4^a}{S(2)S(3)S(4)} < \frac{4^a}{p_1 p_2} < \frac{p_3^a}{p_1 p_2} \]
\[ \frac{5^a}{S(2)S(3)S(4)S(5)} < \frac{5^a}{p_1 p_2 p_3} < \frac{p_4^a}{p_1 p_2 p_3} \]
\[ \frac{6^a}{S(2)S(3)S(4)S(5)S(6)} < \frac{6^a}{p_1 p_2 p_3} < \frac{p_4^a}{p_1 p_2 p_3} \]
\[ \frac{n^a}{S(2)S(3) \cdots S(n)} < \frac{n^a}{p_1 p_2 \cdots p_k} < \frac{p_{k+1}^a}{p_1 p_2 \cdots p_k} \]

where \( p_i \leq n, i \in \{1, \ldots, k\}, p_{k+1} > n \).

Therefore

\[ \sum_{k=1}^{n} \frac{n^a}{S(2)S(3) \cdots S(n)} < 2^{a-1} + \sum_{k=1}^{n} \frac{(p_{k+1} - p_k) \cdot p_{k+1}^a}{p_1 p_2 \cdots p_k} < \]
\[ < 2^{a-1} + \sum_{k=1}^{n} \frac{p_{k+1}^a}{p_1 p_2 \cdots p_k} \]

Then it exists \( k_0 \in \mathbb{N} \) such that for any \( k \geq k_0 \) we have:

\[ p_1 p_2 \cdots p_k > p_{k+1}^{a+1} \]

Therefore

\[ \sum_{k=1}^{n} \frac{n^a}{S(2)S(3) \cdots S(n)} < 2^{a-1} + \sum_{k=1}^{k_0} \frac{p_{k+1}^{a+1}}{p_1 p_2 \cdots p_k} + \sum_{k_0+1}^{k_0} \frac{1}{2^{k+1}} \]
Because the series \( \sum_{k=k_0}^{n} \frac{1}{p_{k+1}} \) is convergent it results that the given series is convergent too.

**Consequence 1.** It exists \( n_0 \in \mathbb{N} \) so that for each \( n \geq n_0 \) we have \( S(2)S(3) \ldots S(n) > n^a \).

**Proof.** Because \( \lim_{n \to \infty} \frac{n^a}{S(2)S(3) \ldots S(n)} = 0 \), there is \( n_0 \in \mathbb{N} \) so that

\[
\frac{n^a}{S(2)S(3) \ldots S(n)} < 1 \text{ for each } n \geq n_0.
\]

**Consequence 2.** It exists \( n_0 \in \mathbb{N} \) so that:

\[
S(2) + S(3) + \ldots + S(n) > (n - 1) \cdot n^{a-1} \text{ for each } n \geq n_0.
\]

**Proof.** We apply the inequality of averages to the numbers \( S(2), S(3), \ldots, S(n) \):

\[
S(2) + S(3) + \ldots + S(n) > (n - 1) \cdot \sqrt[n]{S(2)S(3) \ldots S(n)} > (n - 1) \cdot n^{a-1}, \quad \forall n \geq n_0.
\]

**REFERENCES**


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