ON A SERIES INVOLVING $S(1) \cdot S(2) \cdot \ldots \cdot S(n)$

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For any positive integer $n$ let $S(n)$ be the minimal positive integer $m$ such that $n \mid m!$. It is known that for any $\alpha > 0$, the series
\[
\sum_{n \geq 1} \frac{n^\alpha}{S(1) \cdot S(2) \cdot \ldots \cdot S(n)}
\]
is convergent, although we do not know who was the first to prove the above statement (for example, the authors of [4] credit the paper [1] appeared in 1997, while the result appears also as Proposition 1.6.12 in [2] which was written in 1996).

In this paper we show that, in fact:

**Theorem.**

The series
\[
\sum_{n \geq 1} \frac{x^n}{S(1) \cdot S(2) \cdot \ldots \cdot S(n)}
\]
converges absolutely for every $x$.

**Proof**

Write
\[
a_n = \frac{|x|^n}{S(1) \cdot S(2) \cdot \ldots \cdot S(n)}. \tag{3}
\]

Then
\[
\frac{a_{n+1}}{a_n} = \frac{|x|}{S(n+1)}. \tag{4}
\]

But for $|x|$ fixed, the ratio $|x|/S(n+1)$ tends to zero. Indeed, to see this, choose any positive real number $m$, and let $n_m = \lceil m|x| + 1 \rceil$. When $n > n_m$, it follows that $S(n+1) > \lceil m|x| + 1 \rceil > m|x|$, or $S(n+1)/|x| > m$. Since $m$ was arbitrary, it follows that the sequence $S(n+1)/|x|$ tends to infinity.

**Remarks.**

1. The convergence of (2) is certainly better than the convergence of (1). Indeed, if one fixes any $x > 1$ and any $\alpha$, then certainly $x^n > n^\alpha$ for $n$ large enough.

2. The convergence of (2) combined with the root test imply that
\[(S(1) \cdot S(2) \cdot \ldots \cdot S(n))^{1/n}\]
diverges to infinity. This is equivalent to the fact that the average function of the logs of $S$, namely
\[LS(x) = \frac{1}{x} \sum_{n \leq x} \log S(n) \quad \text{for } x \geq 1\]
tends to infinity with $x$. It would be of interest to study the order of magnitude of the function $LS(x)$. We conjecture that
\[LS(x) = \log x - \log \log x + O(1). \tag{5}\]
The fact that $LS(x)$ cannot be larger than what shows up in the right side of (5) follows from a result from [3]. Indeed, in [3], we showed that

$$A(x) = \frac{1}{x} \sum_{n \leq x} S(n) < 2 \frac{x}{\log x} \quad \text{for} \quad x \geq 64. \quad (6)$$

Now the fact that $LS(x) - \log x + \log \log x$ is bounded above follows from (6) and from Jensen’s inequality for the log function (or the logarithmic form of the AGM inequality). It seems to be considerably harder to prove that $LS(x) - \log x + \log \log x$ is bounded below.

3. As a fun application we mention that for every integer $k \geq 1$, the series

$$\sum_{n \geq 1} \binom{n}{k} \frac{x^n}{S(1) \cdot S(2) \cdot \ldots \cdot S(n)} \quad (7)$$

is absolutely convergent. Indeed, it is a straightforward computation to verify that if one denotes by $C(x)$ the sum of the series (2), then the series (7) is precisely

$$\frac{x^k}{k!} \frac{d^k C}{dx^k}. \quad (8)$$

When $k = x = 1$ series (7) becomes precisely series (1) for $a = 1$.

4. It could be of interest to study the rationality of (2) for integer values of $x$. Indeed, if the function $S$ is replaced with the identity in formula (2), then one obtains the more familiar $e^x$ whose value is irrational (in fact, transcendental) at all integer values of $x$. Is that still true for series (2)?

References