SMARANDACHE FACTOR PARTITIONS OF A TYPICAL CANONICAL FORM.

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ABSTRACT: In [1] we define SMARANDACHE FACTOR PARTITION FUNCTION, as follows:

Let \( \alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r \) be a set of \( r \) natural numbers and \( p_1, p_2, p_3, \ldots p_r \) be arbitrarily chosen distinct primes then \( F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) \) called the Smarandache Factor Partition of \( (\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) \) is defined as the number of ways in which the number

\[
N = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_r^{\alpha_r}
\]

could be expressed as the product of its' divisors. For simplicity, we denote \( F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) = F'(N) \), where

\[
N = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_r^{\alpha_r} \ldots p_n^{\alpha_n}
\]

and \( p_r \) is the \( r^{th} \) prime. \( p_1 = 2, p_2 = 3 \) etc.

In the present note we derive a formula for the case \( N = p_1^{\alpha} p_2^{2} \)

DISCUSSION:

Theorem (5.1):

\[
F'(p_1^{\alpha} p_2^{2}) = F(\alpha,2) = \sum_{k=0}^{\alpha} P(k) + \sum_{j=0}^{\alpha-2} \sum_{i=0}^{i} P(i)
\]

where \( r = [\alpha/2] \) \( \alpha = 2r \) or \( \alpha = 2r + 1 \)
PROOF: Following are the distinct mutually exclusive and exhaustive cases. Only the numbers in the bracket \([\ ]\) are to be further decomposed.

Case I: \((p_2)^{[p_1^\alpha p_2^2]}\) gives \(F^*(p_1^\alpha) = \sum_{k=0}^{\alpha} P(i)\)

Case II: \(\{A_1\} \rightarrow (p_2)^{[p_1^\alpha]} \rightarrow P(\alpha)\)

\(\{A_2\} \rightarrow (p_2^2 p_1^\alpha) \rightarrow P(\alpha-1)\)

\(\ldots\)

\(\{A_\alpha\} \rightarrow (p_2^\alpha p_1^\alpha) \rightarrow P(\alpha-\alpha) = P(0)\)

Hence Case II contributes \(\sum_{i=0}^{\alpha} P(i)\)

Case III: \(\{B_1\} \rightarrow (p_1 p_2) (p_1^2 p_2^2) [p_1^{\alpha-2}] \rightarrow P(\alpha-2)\)

\(\{B_2\} \rightarrow (p_1 p_2) (p_1^2 p_2) [p_1^{\alpha-3}] \rightarrow P(\alpha-3)\)

\(\ldots\)

\(\{B_{\alpha-2}\} \rightarrow (p_1 p_2) (p_1^{\alpha-1} p_2) [p_1^{\alpha-\alpha}] \rightarrow P(\alpha-\alpha) = P(0)\)

Hence Case III contributes \(\sum_{i=0}^{\alpha-2} P(i)\)

Case IV: \(\{C_1\} \rightarrow (p_1^2 p_2) (p_1^2 p_2) [p_1^{\alpha-4}] \rightarrow P(\alpha-4)\)

\(\{C_2\} \rightarrow (p_1^2 p_2) (p_1^3 p_2) [p_1^{\alpha-5}] \rightarrow P(\alpha-5)\)

\(\ldots\)

\(\{C_{\alpha-4}\} \rightarrow (p_1^2 p_2) (p_1^{\alpha-2} p_2) [p_1^{\alpha-\alpha}] \rightarrow P(\alpha-\alpha) = P(0)\)

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Hence Case IV contributes \( \sum_{i=0}^{\alpha-4} P(i) \)

\{ NOTE: The factor partition \((p_1^2 p_2) (p_1 p_2) [p_1^{\alpha-3}] \) has already been covered in case III hence is omitted in case IV. The same logic is extended to remaining (following) cases also. \}

**Case V:** \( \{D_1\} \rightarrow (p_1^3 p_2) (p_1^3 p_2) [p_1^{\alpha-6}] \rightarrow \rightarrow P(\alpha-4) \)

\( \{D_2\} \rightarrow (p_1^3 p_2)(p_1^4 p_2) [p_1^{\alpha-7}] \rightarrow \rightarrow P(\alpha-5) \)

\[ \vdots \]

\( \{D_{\alpha-4}\} \rightarrow (p_1^3 p_2)(p_1^{\alpha-3} p_2) [p_1^{\alpha-\alpha}] \rightarrow \rightarrow P(\alpha-\alpha) = P(0) \)

Hence Case V contributes \( \sum_{i=0}^{\alpha-6} P(i) \)

On similar lines case VI contributes \( \sum_{i=0}^{\alpha-8} P(i) \)

we get contributions upto \( \sum_{i=0}^{\alpha-2r} P(i) \)

where \( 2r < \alpha < 2r +1 \) or \( r = [\alpha/2] \)

summing up all the cases we get

\[ F'(p_1^\alpha p_2^2) = F(\alpha, 2) = \sum_{k=0}^{\alpha} P(k) + \sum_{j=0}^{r} \sum_{i=0}^{\alpha-2j} P(i) \]

where \( r = [\alpha/2] \) \( \alpha = 2r \) or \( \alpha = 2r +1 \)

This completes the proof of theorem (5.1).

**COROLLARY:** (5.1)
\[
F'(p_1^a p_2^2) = \sum_{k=0}^{r} \frac{(k+2) \left[ P(\alpha-2k) + P(\alpha-2k-1) \right]}{k} \quad \text{------(5.1)}
\]

Proof: In theorem (5.1) consider the case \( \alpha = 2r \), we have

\[
F'(p_1^{2r} p_2^2) = F(\alpha,2) = \sum_{k=0}^{2r} P(k) + \sum_{j=0}^{r} \sum_{i=0}^{\alpha-2j} P(i) \quad \text{------(5.2)}
\]

Second term on the RHS can be expanded as follows

\[
P(\alpha) + P(\alpha-1) + P(\alpha-2) + P(\alpha-3) + \ldots + P(2) + P(1) + P(0)
\]

\[
P(\alpha-2) + P(\alpha-3) + \ldots + P(2) + P(1) + P(0)
\]

\[
P(\alpha-4) + \ldots P(2) + P(1) + P(0)
\]

\[
P(\alpha-5) + \ldots + P(2) + P(1) + P(0)
\]

\[
\ldots
\]

\[
P(\alpha-2) + P(\alpha-3) + \ldots + P(2) + P(1) + P(0)
\]

summing up column wise

\[
= \left[ P(\alpha) + P(\alpha-1) \right] + 2 \left[ P(\alpha-2) + P(\alpha-3) \right] + 3 \left[ P(\alpha-4) + P(\alpha-5) \right] + \ldots
\]

\[
+ (r-1) \left[ P(2) + P(1) \right] + r P(0).
\]

\[
= \sum_{k=0}^{r} (k+1) \left[ P(\alpha-2k) + P(\alpha-2k-1) \right]
\]

\[
\text{Here } P(-1) = 0 \text{ has been defined.}
\]

hence

\[
F'(p_1^a p_2^2) = \sum_{k=0}^{r} P(k) + \sum_{k=0}^{r} (k+1) \left[ P(\alpha-2k) + P(\alpha-2k-1) \right]
\]

or

\[
F'(p_1^a p_2^2) = \sum_{k=0}^{r} (k+2) \left[ P(\alpha-2k) + P(\alpha-2k-1) \right]
\]

Consider the case \( \alpha = 2r+1 \), the second term in the expression (5.2) can be expanded as
\( P(\alpha) + P(\alpha-1) + P(\alpha-2) + P(\alpha-3) + \ldots + P(2) + P(1) + P(0) \)
\( P(\alpha-2) + P(\alpha-3) + \ldots + P(2) + P(1) + P(0) \)
\( P(\alpha-4) + \ldots + P(2) + P(1) + P(0) \)
\( P(3) + P(2) + P(1) + P(0) \)
\( P(1) + P(0) \)

summing up column wise we get

\[
= [P(\alpha) + P(\alpha-1)] + 2 [P(\alpha-2) + P(\alpha-3)] + 3 [P(\alpha-4) + P(\alpha-5)] + \ldots \\
+ (r-1) [P(3) + P(2)] + r [P(1) + P(0)].
\]

\[
= \sum_{k=0}^{r} (k+1) [P(\alpha-2k) + P(\alpha-2k-1)], \quad \alpha = 2r+1
\]

on adding the first term, we get

\[
F'(p_1, p_2) = \sum_{k=0}^{[\alpha/2]} (k+2) [P(\alpha-2k) + P(\alpha-2k-1)]
\]

{Note here \( P(-1) \) shall not appear.}
Hence for all values of \( \alpha \) we have

\[
F'(p_1, p_2) = \sum_{k=0}^{[\alpha/2]} (k+2) [P(\alpha-2k) + P(\alpha-2k-1)]
\]

This completes the proof of the Corollary (5.1).

REFERENCES:


[2] "The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texas at Austin, USA.