THE SOLUTION OF THE DIOPHANTINE EQUATION $\sigma_\eta(n) = n$ ($\Omega$)

by Pál Gronás

This problem is closely connected to Problem 29916 in the first issue of the "Smarandache Function Journal" (see page 41 in [1]). The question is: "Are there an infinity of nonprimes $n$ such that $\sigma_\eta(n) = n$?". My calculations will show that the answer is negative.

Let us move on to the first step in deriving the solution of ($\Omega$). As the wording of Problem 29916 indicates. ($\Omega$) is satisfied if $n$ is a prime. This is not the case for $n = 1$ because $\sigma_\eta(1) = 0$.

Suppose $\Pi_{i=1}^k p_i^{r_i}$ is the prime factorization of a composite number $n \geq 4$, where $p_1, \ldots, p_k$ are distinct primes. $r_i \in \mathbb{N}$ and $p_i r_i \geq p_i r_i$ for all $i \in \{1, \ldots, k\}$ and $p_i < p_{i+1}$ for all $i \in \{2, \ldots, k-1\}$ whenever $k \geq 3$.

First of all we consider the case where $k = 1$ and $r_1 \geq 2$. Using the fact that $\eta(p_1^{r_1}) \leq p_1 s_1$ we see that $p_1^{r_1} = n = \sigma_\eta(n) = \sigma_\eta(p_1^{r_1}) = \sum_{d=0}^{r_1} \eta(p_1^d) \leq \sum_{d=0}^{r_1} p_1 s_1 = \frac{p_1 r_1 (r_1 + 1)}{2}$. Therefore $2 p_1^{r_1 - 1} \leq r_1 (r_1 + 1)$ ($\Omega_1$) for some $r_1 \geq 2$. For $p_1 \geq 5$ this inequality ($\Omega_1$) is not satisfied for any $r_1 \geq 2$. So $p_1 < 5$, which means that $p_1 \in \{2, 3\}$. By the help of ($\Omega_1$) we can find a supremum for $r_1$ depending on the value of $p_1$. For $p_1 = 2$ the actual candidates for $r_1$ are 2, 3, 4 and for $p_1 = 3$ the only possible choice is $r_1 = 2$. Hence there are maximum 4 possible solution of ($\Omega$) in this case, namely $n = 4, 8, 9$ and 16. Calculating $\sigma_\eta(n)$ for each of these 4 values, we get $\sigma_\eta(4) = 6$, $\sigma_\eta(8) = 10$, $\sigma_\eta(9) = 9$ and $\sigma_\eta(16) = 16$. Consequently the only solutions of ($\Omega$) are $n = 9$ and $n = 16$.

Next we look at the case when $k \geq 2$:

$$n = \sigma_\eta(n)$$

Substituting $n$ with it's prime factorization we get

$$\prod_{i=1}^k p_i^{r_i} = \sigma_\eta(p_1^{r_1}) = \sum_{d=0}^{r_1} \eta(d) = \sum_{s_1=0}^{r_1} \sum_{s_k=0}^{r_k} \eta(\prod_{i=1}^k p_i^{s_i})$$

$$= \sum_{s_1=0}^{r_1} \sum_{s_k=0}^{r_k} \max\{ \eta(p_1^{s_1}), \ldots, \eta(p_k^{s_k}) \}$$

$$\leq \sum_{s_1=0}^{r_1} \sum_{s_k=0}^{r_k} \max\{ p_1 s_1, \ldots, p_k s_k \} \text{ since } \eta(p_i^{s_i}) \leq p_i s_i$$

$$< \sum_{s_1=0}^{r_1} \sum_{s_k=0}^{r_k} \max\{ p_1 r_1, \ldots, p_k r_k \} \text{ because } s_i \leq r_i$$

$$= \sum_{s_1=0}^{r_1} \sum_{s_k=0}^{r_k} p_1 r_1 (p_1 r_1 \geq p_1 r_i \text{ for } i \geq 2)$$

$$\leq p_1 r_1 \prod_{i=1}^k (r_i + 1).$$
which is equivalent to

\[
\prod_{i=2}^{k} \frac{p_i^r}{r_i + 1} < \frac{p_1 r_1 (r_1 + 1)}{p_1^r} = \frac{r_1 (r_1 + 1)}{p_1^r} \quad (\Omega_2)
\]

This inequality motivates a closer study of the functions \( f(x) = \frac{a^x}{x+1} \) and \( g(x) = \frac{2^x}{x+1} \) for \( x \in [1, \infty) \), where \( a \) and \( b \) are real constants \( \geq 2 \). The derivatives of these two functions are \( f'(x) = \frac{a^x \ln a}{(x+1)^2} [(x+1) \ln a - 1] \) and \( g'(x) = \frac{2^x \ln 2}{(x+1)^2} [(2-\ln 2)x - 1] \). Hence \( f'(x) > 0 \) for \( x \geq 1 \) since \( (x+1) \ln a - 1 \geq (1+1) \ln 2 - 1 = 2 \ln 2 - 1 > 0 \). So \( f \) is increasing on \([1, \infty)\). Moreover \( g(x) \) reaches its absolute maximum value for \( x = \max\{1, \frac{2-\ln 2 + \sqrt{(\ln 2)^2 - 4}}{2 \ln 5} \} \). Now \( \sqrt{(\ln b)^2 + 4} < \ln b + 2 \) for \( b \geq 2 \), which implies that \( \hat{x} = \frac{(2-\ln b + (\ln b + 2)^2)}{2 \ln 5} = \frac{2}{\ln 5} < 2 < 3 \). Furthermore it is worth mentioning that \( f(x) \to \infty \) and \( g(x) \to 0 \) as \( x \to \infty \).

Applying this to our situation means that \( \frac{p_1}{p_1} \) \((i \geq 2)\) is strictly increasing from \( \frac{\hat{x}}{2} \) to \( \infty \). Besides \( \frac{r_1 (r_1 + 1)}{p_1} \leq \max\{2, \frac{4}{p_1}, \frac{12}{p_1^2} \} = \max\{2, \frac{4}{p_1}, \frac{12}{p_1^2} \} \leq 3 \) because \( \frac{4}{p_1} \geq \frac{12}{p_1^2} \) whenever \( p_1 \geq 2 \).

Combining this knowledge with \((\Omega_2)\) we get that \( \Pi_{i=2}^{k} \frac{p_i}{r_i + 1} \leq \Pi_{i=2}^{k} \frac{r_1 (r_1 + 1)}{p_i} \leq \frac{r_1 (r_1 + 1)}{2^k} \leq 3 \) \((\Omega_3)\) for all \( r_1 \in \mathbb{N} \). In other words, \( \Pi_{i=2}^{k} \frac{p_i}{r_i + 1} \leq 3 \). Now \( \Pi_{i=2}^{k} \frac{p_i}{2^k} \leq \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} = \frac{27}{8} > 3 \), which implies that \( k \leq 3 \).

Let us assume \( k = 2 \). Then \((\Omega_2)\) and \((\Omega_3)\) state that \( \frac{p_2}{r_2 + 1} \leq \frac{r_1 (r_1 + 1)}{p_1} \leq \frac{r_1 (r_1 + 1)}{2^2} \) and \( p_2 < 6 \). Next we suppose \( r_2 \geq 3 \). It is obvious that \( p_1 p_2 \geq 2 \cdot 3 = 6 \), which is equivalent to \( p_2 \geq \frac{6}{p_1} \). Using this fact we get \( \frac{p_2}{2^2} \leq \frac{p_2}{r_2 + 1} \leq \frac{r_1 (r_1 + 1)}{p_1} \leq \max\{2, \frac{4}{p_1}, \frac{12}{p_1^2} \} \leq \max\{2, p_2 \} = p_2 \), so \( p_2 \geq 4 \). Accordingly \( p_2 < 2 \), a contradiction which implies that \( r_2 \leq 2 \). Hence \( p_2 \in \{2, 3, 5\} \) and \( r_2 \in \{1, 2\} \).

Furthermore \( 1 \leq \frac{p_2}{r_2 + 1} \leq \frac{r_1 (r_1 + 1)}{p_1} \leq \frac{r_1 (r_1 + 1)}{2^2} \), which implies that \( r_1 \leq 6 \). Consequently, by fixing the values of \( p_2 \) and \( r_2 \), the inequalities \( \frac{r_1 (r_1 + 1)}{p_1} > \frac{p_2}{r_2 + 1} \) and \( p_1 r_1 \geq p_2 r_2 \) give us enough information to determine a supremum (less than 7) for \( r_1 \) for each value of \( p_1 \).

This is just what we have done, and the result is as follows:

<table>
<thead>
<tr>
<th>( p_2 )</th>
<th>( r_2 )</th>
<th>( p_1 )</th>
<th>( r_1 )</th>
<th>( n = p_1^r p_2^s )</th>
<th>( \sigma_n(n) )</th>
<th>IF ( \sigma_n(n) = n ) THEN</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td>1 \leq r_1 \leq 3</td>
<td>2 \cdot 3^1</td>
<td>2 + 3 r_1 (r_1 + 1)</td>
<td>3 \mid 2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>5</td>
<td>1 \leq r_1 \leq 2</td>
<td>2 \cdot 5^1</td>
<td>2 + 5 r_1 (r_1 + 1)</td>
<td>5 \mid 2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>( p_1 \geq 7 )</td>
<td>1</td>
<td>2 p_1</td>
<td>2 + 2 p_1</td>
<td>0 = 2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>36</td>
<td>34</td>
<td>34 = 36</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>( p_1 \geq 5 )</td>
<td>1</td>
<td>4 p_1</td>
<td>3 p_1 + 6</td>
<td>( p_1 = 6 )</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td>2 \leq r_1 \leq 5</td>
<td>3 \cdot 2^2</td>
<td>2 r_1^2 - 2 r_1 + 12</td>
<td>r_1 = 3</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>( p_1 \geq 5 )</td>
<td>1</td>
<td>3 p_1</td>
<td>2 p_1 + 3</td>
<td>( p_1 = 3 )</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>40</td>
<td>30</td>
<td>30 = 40</td>
</tr>
</tbody>
</table>

By looking at the rightmost column in the table above, we see that there are only contradictions except in the case where \( n = 3 \cdot 2^3 \) and \( r_1 = 3 \). So \( n = 3 \cdot 2^3 = 24 \) and \( \sigma_n(24) = 24 \). In other words, \( n = 24 \) is the only solution of \((\Omega)\) when \( k = 2 \).
Finally, suppose \( k = 3 \). Then we know that \( \frac{p_2}{2} \cdot \frac{p_3}{2} < 3 \), i.e. \( p_2 p_3 < 12 \). Hence \( p_2 = 2 \) and \( p_3 \geq 3 \). Therefore \( \frac{r_2(r_2+1)}{2r_2} \leq \frac{r_3(r_3+1)}{2r_3} \leq 2 \) (\( \Omega_4 \)) and by applying (\( \Omega_3 \)) we find that \( p_2^2 \frac{p_3}{2} = \frac{p_2}{2} < 2 \), giving \( p_3 = 3 \).

Combining the two inequalities (\( \Omega_3 \)) and (\( \Omega_4 \)) we get that \( \frac{p_2^2}{2} \cdot \frac{p_3}{2} < 2 \). Knowing that the left side of this inequality is a product of two strictly increasing functions on \([1, \infty)\), we see that the only possible choices for \( r_2 \) and \( r_3 \) are \( r_2 = r_3 = 1 \). Inserting these values in (\( \Omega_3 \)), we get \( \frac{2!}{1!} \cdot \frac{3!}{1!} = 6 < \frac{r_2(r_2+1)}{2r_2} \leq \frac{r_3(r_3+1)}{2r_3} \). This implies that \( r_1 = 1 \). Accordingly (\( \Omega \)) is satisfied only if \( n = 2 \cdot 3 \cdot p_1 = 6 p_1 \):

\[
6 p_1 = \sigma_n(6 p_1) = \eta(1) + \eta(2) + \eta(3) + \eta(6) + \sum_{i=0}^{1} \sum_{j=0}^{1} \eta(2^i 3^j p_1) = 0 + 2 + 3 + 3 + \sum_{i=0}^{1} \sum_{j=0}^{1} \max\{ \eta(p_1), \eta(2^i 3^j) \}
\]

\[
= 8 + \sum_{i=0}^{1} \sum_{j=0}^{1} \max\{ p_1, \eta(2^i 3^j) \}
\]

\[
= 8 + 4 p_1 \text{ because } \eta(2^i 3^j) \leq 3 < p_1 \text{ for all } i, j \in \{0, 1\}
\]

\[
p_1 = 4
\]

which contradicts the fact that \( p_1 \geq 5 \). Therefore (\( \Omega \)) has no solution for \( k = 3 \).

Conclusion: \( \sigma_n(n) = n \) if and only if \( n \) is a prime, \( n = 9, n = 16 \) or \( n = 24 \).

REMARK: A consequence of this work is the solution of the inequality \( \sigma_n(n) > n \) (\( * \)). This solution is based on the fact that (\( * \)) implies (\( \Omega_2 \)).

So \( \sigma_n(n) > n \) if and only if \( n = 8, 12, 18, 20 \) or \( n = 2p \) where \( p \) is a prime. Hence \( \sigma_n(n) \leq n + 4 \) for all \( n \in \mathbb{N} \).

Moreover, since we have solved the inequality \( \sigma_n(n) \geq n \), we also have the solution of \( \sigma_n(n) < n \).

References


Pål Grøndås,
Enges gate 12,
N-7500 Stjørdal,
NORWAY.