THE SOLUTION OF THE DIOPHANTINE EQUATION $\sigma_\eta(n) = n$ (Ω)

by Pål Grønås

This problem is closely connected to Problem 29916 in the first issue of the "Smarandache Function Journal" (see page 47 in [1]). The question is: "Are there an infinity of nonprimes n such that $\sigma_\eta(n) = n$?". My calculations will show that the answer is negative.

Let us move on to the first step in deriving the solution of (Ω). As the wording of Problem 29916 indicates, (Ω) is satisfied if n is a prime. This is not the case for $n = 1$ because $\sigma_\eta(1) = 0$.

Suppose $\prod_{i=1}^{k} p_i^{r_i}$ is the prime factorization of a composite number $n \geq 4$, where $p_1, \ldots, p_k$ are distinct primes, $r_i \in \mathbf{N}$ and $p_1 r_1 \geq p_i r_i$ for all $i \in \{1, \ldots, k\}$ and $p_i < p_{i+1}$ for all $i \in \{2, \ldots, k-1\}$ whenever $k \geq 3$.

First of all we consider the case where $k = 1$ and $r_1 \geq 2$. Using the fact that $\eta(p_1^{s_1}) \leq p_1 s_1$ we see that $p_1^{r_1} = n = \sigma_\eta(n) = \sigma_\eta(p_1^{r_1}) = \sum_{s_1=0}^{r_1} \eta(p_1^{s_1}) \leq \sum_{s_1=0}^{r_1} p_1 s_1 = \frac{p_1 r_1 (r_1+1)}{2}$. Therefore $2 p_1^{r_1 - 1} \leq r_1(r_1+1)$ (Ω_1) for some $r_1 \geq 2$. For $p_1 \geq 5$ this inequality (Ω_1) is not satisfied for any $r_1 \geq 2$. So $p_1 < 5$, which means that $p_1 \in \{2, 3\}$. By the help of (Ω_1) we can find a supremum for r_1 depending on the value of p_1. For $p_1 = 2$ the actual candidates for r_1 are 2, 3, 4 and for $p_1 = 3$ the only possible choice is $r_1 = 2$. Hence there are maximum 4 possible solution of (Ω) in this case, namely $n = 4$, 8, 9 and 16. Calculating $\sigma_\eta(n)$ for each of these 4 values, we get $\sigma_\eta(4) = 6$, $\sigma_\eta(8) = 10$, $\sigma_\eta(9) = 9$ and $\sigma_\eta(16) = 16$. Consequently the only solutions of (Ω) are $n = 9$ and $n = 16$.

Next we look at the case when $k \geq 2$:

$$n = \sigma_\eta(n)$$

Substituting n with it's prime factorization we get

$$\begin{aligned}
\prod_{i=1}^{k} p_i^{r_i} &= \sigma_\eta\Big(\prod_{i=1}^{k} p_i^{r_i}\Big) = \sum_{\substack{d \mid n \\ d>0}} \eta(d) = \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} \eta\Big(\prod_{i=1}^{k} p_i^{s_i}\Big) \\
&= \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} \max\{\, \eta(p_1^{s_1}), \ldots, \eta(p_k^{s_k}) \,\} \\
&\leq \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} \max\{\, p_1 s_1, \ldots, p_k s_k \,\} \quad \text{since} \quad \eta(p_i^{s_i}) \leq p_i s_i \\
&< \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} \max\{\, p_1 r_1, \ldots, p_k r_k \,\} \quad \text{because} \quad s_i \leq r_i \\
&= \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} p_1 r_1 \quad (p_1 r_1 \geq p_i r_i \text{ for } i \geq 2) \\
&\leq p_1 r_1 \prod_{i=1}^{k} (r_i + 1),
\end{aligned}$$

which is equivalent to

$$\prod_{i=2}^{k} \frac{p_i^{r_i}}{r_i+1} < \frac{p_1 r_1(r_1+1)}{p_1^{r_1}} = \frac{r_1(r_1+1)}{p_1^{r_1-1}} \quad (\Omega_2)$$

This inequality motivates a closer study of the functions $f(x) = \frac{a^x}{x+1}$ and $g(x) = \frac{x(x+1)}{b^x-1}$ for $x \in [1, \infty)$, where a and b are real constants ≥ 2. The derivatives of these two functions are $f'(x) = \frac{a^x}{(x+1)^2}[(x+1)\ln a - 1]$ and $g'(x) = \frac{(-\ln b)x^2+(2-\ln b)x+1}{b^x-1}$. Hence $f'(x) > 0$ for $x \geq 1$ since $(x+1)\ln a - 1 \geq (1+1)\ln 2 - 1 = 2\ln 2 - 1 > 0$. So f is increasing on $[1, \infty)$. Moreover $g(x)$ reaches its absolute maximum value for $x = \max\{1, \frac{2-\ln b+\sqrt{(\ln b)^2+4}}{2\ln b} = \hat{x}\}$. Now $\sqrt{(\ln b)^2+4} < \ln b + 2$ for $b \geq 2$, which implies that $\hat{x} < \frac{(2-\ln b)+(\ln b+2)}{2\ln b} = \frac{2}{\ln b} \leq \frac{2}{\ln 2} < 3$. Futhermore it is worth mentioning that $f(x) \to \infty$ and $g(x) \to 0$ as $x \to \infty$.

Applying this to our situation means that $\frac{p_i^{r_i}}{r_i+1}$ ($i \geq 2$) is strictly increasing from $\frac{p_i}{2}$ to ∞. Besides $\frac{r_1(r_1+1)}{p_1^{r_1-1}} \leq \max\{2, \frac{6}{p_1}, \frac{12}{p_1^2}\} = \max\{2, \frac{6}{p_1}\} \leq 3$ because $\frac{6}{p_1} \geq \frac{12}{p_1^2}$ whenever $p_1 \geq 2$. Combining this knowledge with (Ω_2) we get that $\prod_{i=2}^{k} \frac{p_i}{2} \leq \prod_{i=2}^{k} \frac{p_i^{r_i}}{r_i+1} < \frac{r_1(r_1+1)}{p_1^{r_1-1}} \leq \frac{r_1(r_1+1)}{2^{r_1-1}} \leq 3$ (Ω_3) for all $r_1 \in \mathbb{N}$. In other words, $\prod_{i=2}^{k} \frac{p_i}{2} < 3$. Now $\prod_{i=2}^{4} \frac{p_i}{2} \geq \frac{2}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} = \frac{15}{4} > 3$, which implies that $k \leq 3$.

Let us assume $k = 2$. Then (Ω_2) and (Ω_3) state that $\frac{p_2^{r_2}}{r_2+1} < \frac{r_1(r_1+1)}{p_1^{r_1-1}}$ and $\frac{p_2}{2} < 3$, i.e. $p_2 < 6$. Next we suppose $r_2 \geq 3$. It is obvious that $p_1 p_2 \geq 2 \cdot 3 = 6$, which is equivalent to $p_2 \geq \frac{6}{p_1}$. Using this fact we get $\frac{p_2^3}{4} \leq \frac{p_2^{r_2}}{r_2+1} < \frac{r_1(r_1+1)}{p_1^{r_1-1}} \leq \max\{2, \frac{6}{p_1}\} \leq \max\{2, p_2\} = p_2$, so $p_2^2 < 4$. Accordingly $p_2 < 2$, a contradiction which implies that $r_2 \leq 2$. Hence $p_2 \in \{2, 3, 5\}$ and $r_2 \in \{1, 2\}$.

Futhermore $1 \leq \frac{p_2}{2} \leq \frac{p_2^{r_2}}{r_2+1} < \frac{r_1(r_1+1)}{p_1^{r_1-1}} \leq \frac{r_1(r_1+1)}{2^{r_1-1}}$, which implies that $r_1 \leq 6$. Consequently, by fixing the values of p_2 and r_2, the inequalities $\frac{r_1(r_1+1)}{p_1^{r_1-1}} > \frac{p_2^{r_2}}{r_2+1}$ and $p_1 r_1 \geq p_2 r_2$ give us enough information to determine a supremum (less than 7) for r_1 for each value of p_1.

This is just what we have done, and the result is as follows:

p_2	r_2	p_1	r_1	$n = p_1^{r_1} p_2^{r_2}$	$\sigma_\eta(n)$	IF $\sigma_\eta(n) = n$ THEN
2	1	3	$1 \leq r_1 \leq 3$	$2 \cdot 3^{r_1}$	$2 + 3r_1(r_1+1)$	$3 \mid 2$
2	1	5	$1 \leq r_1 \leq 2$	$2 \cdot 5^{r_1}$	$2 + 5r_1(r_1+1)$	$5 \mid 2$
2	1	$p_1 \geq 7$	1	$2p_1$	$2 + 2p_1$	$0 = 2$
2	2	3	2	36	34	$34 = 36$
2	2	$p_1 \geq 5$	1	$4p_1$	$3p_1 + 6$	$p_1 = 6$
3	1	2	$2 \leq r_1 \leq 5$	$3 \cdot 2^{r_1}$	$2r_1^2 - 2r_1 + 12$	$r_1 = 3$
3	1	$p_1 \geq 5$	1	$3p_1$	$2p_1 + 3$	$p_1 = 3$
5	1	2	3	40	30	$30 = 40$

By looking at the rightmost column in the table above, we see that there are only contradictions except in the case where $n = 3 \cdot 2^{r_1}$ and $r_1 = 3$. So $n = 3 \cdot 2^3 = 24$ and $\sigma_\eta(24) = 24$. In other words, $n = 24$ is the only solution of (Ω) when $k = 2$.

Finally, suppose $k = 3$. Then we know that $\frac{p_2}{2} \cdot \frac{p_3}{2} < 3$, i.e. $p_2 p_3 < 12$. Hence $p_2 = 2$ and $p_3 \geq 3$. Therefore $\frac{r_1(r_1+1)}{p_1^{r_1-1}} \leq \frac{r_1(r_1+1)}{3^{r_1-1}} \leq 2$ (Ω_4) and by applying (Ω_3) we find that $\prod_{i=2}^{3} \frac{p_i}{2} = \frac{p_3}{2} < 2$, giving $p_3 = 3$.

Combining the two inequalities (Ω_2) and (Ω_4) we get that $\frac{2^{r_2}}{r_2+1} \cdot \frac{3^{r_3}}{r_3+1} < 2$. Knowing that the left side of this inequality is a product of two strictly increasing functions on $[1, \infty)$, we see that the only possible choices for r_2 and r_3 are $r_2 = r_3 = 1$. Inserting these values in (Ω_2), we get $\frac{2^1}{1+1} \cdot \frac{3^1}{1+1} = \frac{3}{2} < \frac{r_1(r_1+1)}{p_1^{r_1-1}} \leq \frac{r_1(r_1+1)}{5^{r_1-1}}$. This implies that $r_1 = 1$. Accordingly (Ω) is satisfied only if $n = 2 \cdot 3 \cdot p_1 = 6\,p_1$:

$$
\begin{aligned}
6\,p_1 &= \sigma_\eta(6\,p_1) \\
&= \eta(1) + \eta(2) + \eta(3) + \eta(6) + \sum_{i=0}^{1}\sum_{j=0}^{1} \eta(2^i\,3^j\,p_1) \\
&= 0 + 2 + 3 + 3 + \sum_{i=0}^{1}\sum_{j=0}^{1} \max\{\,\eta(p_1), \eta(2^i\,3^j)\,\} \\
&= 8 + \sum_{i=0}^{1}\sum_{j=0}^{1} \max\{\,p_1, \eta(2^i\,3^j)\,\} \\
&= 8 + 4\,p_1 \quad \text{because} \quad \eta(2^i\,3^j) \leq 3 < p_1 \quad \text{for all} \quad i, j \in \{0, 1\} \\
&\Downarrow \\
p_1 &= 4
\end{aligned}
$$

which contradicts the fact that $p_1 \geq 5$. Therefore (Ω) has no solution for $k = 3$.

Conclusion: $\sigma_\eta(n) = n$ if and only if n is a prime, $n = 9$, $n = 16$ or $n = 24$.

REMARK: A consequence of this work is the solution of the inequality $\sigma_\eta(n) > n$ $(*)$. This solution is based on the fact that $(*)$ implies (Ω_2).

So $\sigma_\eta(n) > n$ if and only if $n = 8, 12, 18, 20$ or $n = 2p$ where p is a prime. Hence $\sigma_\eta(n) \leq n + 4$ for all $n \in \mathbf{N}$.

Moreover, since we have solved the inequality $\sigma_\eta(n) \geq n$, we also have the solution of $\sigma_\eta(n) < n$.

References

[1] Smarandache Function Journal, Number Theory Publishing Co., Phoenix, New York, Lyon, Vol. 1, No. 1, 1990.

Pål Grønås,
Enges gate 12,
N-7500 Stjørdal,
NORWAY.