THE SEMILATTICE WITH CONSISTENT RETURN

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Let \( p \) be a prime number. In [5] is defined the function \( S_p: N^* \to N^* \), \( S_p(a) = k \), where \( k \) is the smallest positive integer so that \( p^a \) is a divisor for \( k! \).

A Smarandache function of first kind is defined for each \( n \in N^* \) in [1], as numerical function \( S_n: N^* \to N^* \), so that:

i) if \( n = u^a \), where \( u = 1 \) or \( u = p \), then \( S_n(a) = k \), \( k \) being the smallest positive integer with the property that \( k! = M \cdot u^a \).

ii) if \( n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_r^{a_r} \), then \( S_n(a) = \max_{1 \leq j \leq r} \left\{ S_{p_j}(i) \cdot \frac{a_j}{a_j} \right\} \).

It is proved that:

\[
\sum_{1} \max\{S_n(a), S_n(b)\} \leq S_n(a + b) \leq S_n(a) + S_n(b)
\]

\[
\sum_{2} \quad S_n(a + b) \leq S_n(a) \cdot S_n(b)
\]

In [2] is proved that:

i) the function \( S_n \) is monotonously increasing,

ii) the sequence of functions \( \{S_{p^i}\}_{i \in N^*} \) is monotonously increasing.

iii) for \( p, q \) - prime numbers such that: \( p < q \Rightarrow S_p < S_q \) and \( p \cdot i < q \Rightarrow S_{p^i} < S_q \), where \( i \in N^* \).

iv) if \( n < p \), then \( S_n < S_p \).

In [3] it is proved:

i) for \( p \geq 5 \), \( S_p > \max\{S_{p-1}, S_{p+1}\} \)

ii) for \( p, q \) - prime numbers, \( i, j \in N^* \)

\[
p < q \quad \text{and} \quad i < j \quad \Rightarrow \quad S_{p^i} < S_{q^j}
\]

iii) the sequence of functions \( \{S_n\}_{n \in N^*} \) is generally increasing bounded

iv) if \( n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_r^{k_r} \), there are \( k_1, k_2, \ldots, k_m \in \{1, 2, \ldots, r\} \) so that for each \( t \in \overline{1, m} \) there is \( q_t \in N^* \) so that

\[
S_n(q_t) = S_{p_{k_t}}(q_t)
\]
and for each \( l \in N^* \) we have:

\[
S_n(l) = \max_{1 \leq i \leq m} \left\{ S_{p_i^{k_i}}(l) \right\}.
\]

We define the set \( \left\{ p_i^{k_i} \mid t \in \{1, m\} \right\} \) as the set of active factors of \( n \) and the others factors as the passive factors.

Let \( N_{p_1, p_2, \ldots, p_r} = \left\{ n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_r^{k_r} \mid i_1, i_2, \ldots, i_r \in N^* \right\} \), where \( p_1 < p_2 < \cdots < p_r \) are prime numbers. Then

\[
N_{p_1, p_2, \ldots, p_r} = \left\{ n \in N_{p_1, p_2, \ldots, p_r} \mid n \text{ has } p_1^{k_1}, p_2^{k_2}, \ldots, p_r^{k_r} \text{ as active factors} \right\}
\]

is the S-active cone.

A Smarandache function of second kind is defined for each \( k \in N^* \) in [1], as the function \( S^k : N^* \rightarrow N^* \) where \( S^k(n) = S_n(k) \).

It is proved that:

\[
\sum_{3} \text{max}\left\{ S^k(a), S^k(b) \right\} \leq S^k(a \cdot b) \leq S^k(a) + S^k(b)
\]

\[
\sum_{4} S^k(a \cdot b) \leq S^k(a) \cdot S^k(b)
\]

In [4] it is proved that:

i) for \( k, n \in N^* \) the formula \( S^k(n) \leq n \cdot k \) is true

ii) all prime numbers \( p \geq 5 \) are maximal points for \( S^k \) and

\[
S^k(p) = p[k - i_p(k)], \text{ where } 0 \leq i_p(k) \leq \left\lfloor \frac{k-1}{p} \right\rfloor
\]

iii) the function \( S^k \) has its relative minimum values for every \( n = p! \), where \( p \) is a prime number and \( p \geq \max\{3, k\} \)

iv) the numbers \( kp \) for \( p \) prime number, \( k \in N^* \) and \( p > k \), are the fixed points of \( S^k \)

v) the function \( S^k \) have the following properties:

a) \( S^k = 0 \) \( (n^{1+\varepsilon}) \), for \( \varepsilon > 0 \)

b) \( \lim_{n \to \infty} \sup \frac{S^k(n)}{n} = k \)

c) \( S^k \) is, "generally speaking", increasing, thus:
1. **DEFINITION.** Let $\mathcal{M} = \{S_m(n)|n,m \in \mathbb{N}^*\}$, let $A, B \in \mathcal{P}(\mathbb{N}^*) \setminus \emptyset$ and $a = \min A$, $b = \min B, a^* = \max A, b^* = \max B$. The set $I$ is the set of the functions:

$$I_A^B : \mathbb{N}^* \rightarrow \mathcal{M}, \text{ with } I_A^B(n) = \begin{cases} S_a(b), n < \max\{a, b\} \\ S_{a_k}(b_k), \max\{a, b\} \leq n \leq \max\{a^k, b^k\} \\ \text{ where } \\ a_k = \max \{a_i \in A | a_i \leq n\} \\ b_k = \max \{b_j \in B | b_j \leq n\} \\ S_{a^*}(b^*), n > \max\{a^*, b^*\} \end{cases}$$

2. **EXAMPLES.**

a) $I_{\{6,10,12\}}^{\{3,8,10\}} : \mathbb{N}^* \rightarrow \mathcal{M}$ and:

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>$n \geq 13$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_{{6,10,12}}^{{3,8,10}}$</td>
<td>$S_3(6)$</td>
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<td>$S_3(6)$</td>
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<td>$S_{10}(12)$</td>
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b) Let $A = \{1,3,5,...,2k+1,...\}$

$B = \{2,4,6,...,2k,...\}$

$J_A^B : \mathbb{N}^* \rightarrow \mathcal{M}$ and:

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>...</th>
<th>2k</th>
<th>2k+1</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_A^B$</td>
<td>$S_2(1)$</td>
<td>$S_2(1)$</td>
<td>$S_2(3)$</td>
<td>$S_4(3)$</td>
<td>$S_4(5)$</td>
<td>$S_6(5)$</td>
<td>...</td>
<td>$S_{2k}(2k-1)$</td>
<td>$S_{2k}(2k+1)$</td>
<td>...</td>
</tr>
</tbody>
</table>

c) Let $A = \{5,9,10\}$ and $J_A^B, I_{\mathbb{N}^*}^N : \mathbb{N}^* \rightarrow \mathcal{M}$ with
It is easy to see that $I_A^A$ is not the reduction of $I_N^{N^*}$ and $I_A^A(N^*) \subseteq I_N^{N^*}(N^*)$.

3. REMARK.

The functions which belong to the set $I$ have the following properties:

1) if $A_1 \subseteq A_2$ and $n \in A_1$, then $I_A^B(n) = I_{A_2}^B(n)$

1') if $B_1 \subseteq B_2$ and $n \in B_1$, then $I_A^B(n) = I_{A_2}^B(n)$

2) $I_N^{N^*}(n) = S_A(n) = S_B(n)$, the function $I_N^{N^*}$ is called the I-diagonal function and $I_N^{N^*}(N^*)$ is called the diagonal of $\mathcal{A}$.

3) for each $m \in N^* I_N^{N^*}_{\{m\}} = S_m$ for $I_N^{N^*}_{\{m\}}(n) = S_m(n), \forall n \in N^*$.

3') for each $m \in N^* I_N^{N^*}_{\{m\}} = S_m$ for $I_N^{N^*}_{\{m\}}(n) = S_m(n), \forall n \in N^*$.

4) if $n \in A \cap B$, then $I_A^B(n) = I_{A_2}^B(n)$.

4. DEFINITION. For each pair $m, n \in N^*$, $S_m(n)$ and $S(m, n)$ are called the symmetrical numbers relative to the diagonal of $\mathcal{A}$.

$S_m$ and $S(m, n)$ are called the symmetrical functions relative to the I-diagonal function $I_N^{N^*}$. As a rule, $I_A^B$ and $I_A^A$ are called the symmetrical functions relative to the I-diagonal function $I_N^{N^*}$.

5. DEFINITION. Let us consider the following rule $\mathcal{T}: I \times I \rightarrow I$, $I_A^B \mathcal{T} I_C^D = I_A^{B \cup D \cap C}$. It is easy to see that $\mathcal{T}$ is idempotent, commutative and associative, so that:

i) $I_A^B \mathcal{T} I_A^B = I_A^B$

ii) $I_A^B \mathcal{T} I_C^D = I_C^D \mathcal{T} I_A^B$

iii) $(I_A^B \mathcal{T} I_C^D) \mathcal{T} I_E^F = I_A^B \mathcal{T} (I_C^D \mathcal{T} I_E^F)$, where $A, B, C, D, E, F \in \mathcal{P}(N^*) \setminus \emptyset$

6. DEFINITION. Let us consider the following relative partial order relation $\mathcal{P}$, where:

$$\mathcal{P} \subset I \times I,$$

$$I_A^B \mathcal{P} I_C^D \Rightarrow A \subset C \text{ and } B \subset D.$$ 

It is easy to see that $(I, \mathcal{T}, \mathcal{P})$ is a semilattice.
7. DEFINITION. The elements \( u, v \in I \) are \( \rho \)-preceded if there is \( w \in I \) so that:

\[
w \rho u \text{ and } w \rho v.
\]

8. DEFINITION. The elements \( u, v \in I \), are \( \rho \)-strictly preceded by \( w \) if:

i) \( w \rho u \) and \( w \rho v \).

ii) \( \forall x \in I \setminus \{w\} \) so that \( x \rho u \) and \( x \rho v \Rightarrow x \rho w \).

9. DEFINITION. Let us defined:

\[
I^* = \{(u, v) \in I \times I | u, v \text{ are } \rho \text{-preceded}\}
\]

\[
I^# = \{(u, v) \in I \times I | u, v \text{ are } \rho \text{-strictly preceded}\}.
\]

It is evidently that \( (u, v) \in I^* \Leftrightarrow (v, u) \in I^* \) and \( (u, v) \in I^# \Leftrightarrow (v, u) \in I^# \).

10. DEFINITION. Let us consider \( T^# = U \times U \), \( U \subseteq I \) and let us consider the following rule:

\[
\downarrow : I^# \to W, \ W \subseteq I, \ I_A \bot I_C = I_B \land D \quad \text{and the ordering partial relation } r \subseteq U \times U \quad \text{so that}
\]

\[
I^B_A \cap I_D^C \Rightarrow I^C_D \rho I^B_A.
\]

The structure \((I^#, \bot, r)\) is called the return of semilattice \((I, T, \rho)\).

11. DEFINITION. The following set

\[
\mathcal{B} = \{I^B_A \in I | A \cap B \neq \emptyset\}
\]

is called the base of return \((I^#, \bot, r)\).

12. REMARK. The base of return has the following properties:

i) if \( I^B_A \in \mathcal{B} \Rightarrow I^A_B \in \mathcal{B} \)

ii) for \( \emptyset \neq X \subset N^*, I^X_C \in \mathcal{B} \)

iii) for \( I^B_A \in \mathcal{B} \) is true the following equivalence \( \emptyset \neq X \subseteq C_N^*(A \land B) \Leftrightarrow \text{non existence of } I^X_C \perp I^B_A \).

13. PROPOSITION. For \( I^B_A \in \mathcal{B} \) there exists \( n \in N^* \) so that \( I^B_A(n) = I^{N^*}(n) \).

Proof. Because \( A \cap B \neq \emptyset \) it results that there exists \( n \in A \cap B \) so that:

\[
I^B_A(n) = S_n(n) = I^{N^*}_N(n).
\]

It results that for \( I^B_A \in \mathcal{B} \) then \( I^B_A \) has at least a point of contact with \( I \)-diagonal function.
14. **REMARK.** From the 1. it results:

\[ I^B_{\{n\}}(n) = S_n(b_n), \text{ where } b_n = \begin{cases} b, & n < b = \min B \\ b_k, & b \leq n \leq b^* = \max B \\ b_k = \max\{x \in B | x \leq n\} \\ b^*, & n > b^* \end{cases} \]

and

\[ I^{(m)}_{A}(m) = S^m(a_m), \text{ where } a_m = \begin{cases} a, & m < a = \min A \\ a_k, & a \leq m \leq a^* = \max A \\ a_k = \max\{x \in A | x \leq m\} \\ a^*, & m > a^* \end{cases} \]

15. **PROPOSITION.** There are true the following equivalences:

\[ \left( I^B_A, I^D_C \right) \in I^\# \iff I^C_A, I^D_B \in \mathcal{B} \iff \exists n, m \in \mathbb{N}^* \text{ so that:} \]

\[ I^B_A(n) = I^B_{\{n\}}(n) = S_n(b_n), \quad I^D_A(n) = I^D_{\{n\}}(n) = S_n(d_n), \quad I^B_A(m) = I^{(m)}_{A}(m) = S^m(a_m), \quad \text{and} \]

\[ I^D_C(m) = I^{(m)}_{C}(m) = S^m(c_m) \text{ where } a_m, b_n, c_m, d_n \text{ are defined in the sense of 14.} \]

If \( n \leq m \), then \( n \leq a_m, c_m \leq m \).

**Proof.** Evidently,

\[ \left( I^B_A, I^D_C \right) \in I^\# \iff A \cap C \neq \emptyset \text{ and } B \cap D \neq \emptyset \iff I^C_A, I^D_B \in \mathcal{B}. \]

Because \( A \cap C \neq \emptyset \) and \( B \cap D \neq \emptyset \) it exists \( n \in A \cap C \) and \( m \in B \cap D \). Then:

\[ I^B_A(n) = I^B_{\{n\}}(n) = S_n(b_n), \quad I^D_A(n) = I^D_{\{n\}}(n) = S_n(d_n) \]

\[ I^B_A(m) = I^{(m)}_{A}(m) = S^m(a_m), \quad I^D_A(m) = I^{(m)}_{A}(m) = S^m(c_m). \]

Conversely, if there exist \( n \in \mathbb{N}^* \) so that \( I^B_A(n) = S_n(b_n) \) and \( I^D_C(n) = S_n(d_n) \), then because \( I^B_A(n) = S_n(b_n) \) it results \( n = a_k = \max\{a_i \in A | a_i \leq n\} \), so that \( n \in A \). Because \( I^D_C(n) = S_n(d_n) \) it results \( n \in C \).

Therefore \( A \cap C \neq \emptyset \), thus, finally, \( I^C_A \in \mathcal{B} \). It is also proved \( I^D_B \in \mathcal{B} \) in the same way.

If \( n \leq m \), because \( n \in A \cap C \) it results that \( n \in \{x \in A | x \leq m\} \) and \( n \in \{y \in C | y \leq m\} \), therefore \( n \leq a_m \leq m \) and \( n \leq c_m \leq m \).

156
16. DEFINITION. The return \((I^\#, \perp, r)\) of semilattice \((L, T, \rho)\) is:

- a) null, if \(L^\# = \{(u, u) | u \in L\} = \Delta_L\).
- b) weak, if \(\text{card}(L^\#) < \text{card}(L \times L \setminus I^\#)\).
- c) consistent, if \(\text{card}(L^\#) = \text{card}(L \times L \setminus I^\#)\).
- d) vigour, if \(\text{card}(L^\#) > \text{card}(L \times L \setminus I^\#)\).
- e) total, if \(L^\# = L \times L\).

17. PROPOSITION. The return \((I^\#, \perp, r)\) of the semilattice \((I, T, \rho)\) is consistent.

Proof. Evidently, \(\text{card}(\mathcal{P}(N^*) \setminus \emptyset) = \aleph\), \(\text{card} I = \text{card}[\mathcal{P}(N^*) - \emptyset] = \aleph\) and \(\text{card}(I \times I) = \aleph\).

Let us consider \(\mathcal{F} = \{(A, C) | A, C \in \mathcal{P}(N^*) - \emptyset, A \cap C = \emptyset\}\) and \(\overline{\mathcal{F}} = \{(A, C) | A, C \in \mathcal{P}(N^*) - \emptyset, A \cap C \neq \emptyset\}\).

\(\text{card}(\mathcal{F}) = \text{card}(\overline{\mathcal{F}}) = \aleph\). Indeed, if \(A \cap C = \emptyset\) it results that \(C_{N^*} \cdot A \cup C_{N^*} \cdot C = N^*\); because for every \(X \in P(N^*) - \emptyset\) \(\exists Y = N^* \setminus X\) so that \(X \cup Y = N^*\) then it results \(\text{card}(\mathcal{F}) = \text{card}(\mathcal{P}(N^*)) = \aleph\). Because for each \((A, C), A, C \in \mathcal{P}(N^*) - \emptyset, A \cap C = \emptyset\), it exist at least two \((A_1, C_1), (A_2, C_2)\) with \(A_1 \cap C_1 \neq \emptyset, A_2 \cap C_2 \neq \emptyset\) it results \(\text{card}(\overline{\mathcal{F}}) \geq \text{card}(\mathcal{F}) = \aleph\).
Since \( \text{card} \mathcal{F} \leq \text{card} [\mathcal{P}(N^*) - \emptyset] = N \) finally \( \text{card} \mathcal{F} = N \). Because \( \text{card} f^# = \text{card} (\mathcal{F} \times \mathcal{F}) = N \) and \( \text{card} (I \times I) - I^# = \text{card} (\mathcal{F} \times \mathcal{F}) = N \) it results that \( (f^#, \bot, r) \) is a return consistent.

18. REMARK. Generally, it is interesting the following problems:

i) what relations, operations, structures can be defined on

\[
M = \left\{ S_m(n) \mid n, m \in N^* \right\}
\]

ii) what relations, operations, structures can be defined on

\[ \mathcal{H} = \{ f \mid f : N^* \rightarrow \mathcal{H} \} \]

REFERENCES