Given a positive integer \( n \), let \( P(n) \) denote the largest prime factor of \( n \) and \( S(n) \) denote the smallest integer \( m \) such that \( n \) divides \( m! \).

This paper extends earlier work [1] on the average value of the Smarandache function \( S(n) \) and is based on a recent asymptotic result [2]:

\[
\left| \{ n \leq N : P(n) < S(n) \} \right| = o \left( \frac{N}{\ln(N)^j} \right)
\]

for any positive integer \( j \) due to Ford. We first prove:

**Theorem 1.**

\[
E(S(N)^k) = \frac{1}{N} \sum_{n=1}^{N} S(n)^k = \frac{\zeta(k+1)}{k+1} \cdot \frac{N^k}{\ln(k)} + O \left( \frac{N^k}{\ln(N)^2} \right)
\]

where \( \zeta(x) \) is the Riemann zeta function. In particular,

\[
\lim_{N \to \infty} \frac{\ln(N)}{N} \cdot E(S(N)) = \frac{\pi^2}{12} = 0.82246703...
\]

\[
\lim_{N \to \infty} \frac{\ln(N)}{N^2} \cdot \text{Var}(S(N)) = \frac{\zeta(3)}{3} = 0.40068563...
\]

**Sketch of Proof.** On one hand,

\[
L(k) = \lim_{n \to \infty} \frac{\ln(n)}{n^k} \cdot E(P(n)^k) \leq \lim_{n \to \infty} \frac{\ln(n)}{n^k} \cdot E(S(n)^k) = \lim_{N \to \infty} \frac{\ln(N)}{N^{k+1}} \cdot \sum_{n=1}^{N} S(n)^k
\]

The above summation, on the other hand, breaks into two parts:

\[
\lim_{N \to \infty} \frac{\ln(N)}{N^{k+1}} \cdot \left( \sum_{P(n)=S(n)} P(n)^k + \sum_{P(n)<S(n)} S(n)^k \right)
\]
The second part vanishes:

\[
\lim_{N \to \infty} \ln(N) \cdot \left( \sum_{P(n) < S(n)} \left( \frac{S(n)}{N} \right)^k \right) \leq \lim_{N \to \infty} \ln(N) \cdot \left( \sum_{P(n) < S(n)} \frac{1}{N} \cdot \frac{N}{\ln(N)} \right) = 0
\]

while the first part is bounded from above:

\[
\lim_{N \to \infty} \frac{\ln(N)}{N^{k+1}} \cdot \left( \sum_{P(n) = S(n)} P(n)^k \right) \leq \lim_{N \to \infty} \frac{\ln(N)}{N^{k+1}} \cdot \sum_{n=1}^{N} P(n)^k = \lim_{n \to \infty} \frac{\ln(n)}{n^k} \cdot E(P(n)^k) = L(k)
\]

A formula for \( L(k) \) was found by Knuth and Trabb Pardo [3] and the remaining second-order details follow similarly.

Observe that the ratio \( \sqrt{\text{Var}(S(N))} / E(S(N)) \to \infty \text{ as } N \to \infty \), which indicates that the traditional sample moments are unsuitable for estimating the probability distribution of \( S(N) \). An alternative estimate involves the relative number of digits in the output of \( S \) per digit in the input. A proof of the following is similar to [1]; the integral formulas were discovered by Shepp and Lloyd [4].

**Theorem 2.**

\[
\lim_{N \to \infty} E \left( \left( \frac{\ln(S(N))}{\ln(N)} \right)^k \right) = \int_0^\infty \frac{x^{k-1}}{k!} \cdot \exp \left( -x - \int_x^\infty \frac{e^{-y}}{y} \, dy \right) \, dx =
\]

\[
\begin{cases}
0.62432998 & \text{if } k = 1 \\
0.42669576 & \text{if } k = 2 \\
0.31363067 & \text{if } k = 3 \\
0.24387660 & \text{if } k = 4 \\
0.19792289 & \text{if } k = 5
\end{cases}
\]

The mean output of \( S \) hence has asymptotically 62.43\% of the number of digits of the input, with a standard deviation of 19.21\%. A web-based essay on the Golomb-Dickman constant 0.62432998... appears in [5] and has further extensions and references.

**References**