THE MONOTONY OF SMARANDACHE FUNCTIONS
OF FIRST KIND

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Smarandache functions of first kind are defined in [1] thus:

\[ S_n : \mathbb{N}^* \rightarrow \mathbb{N}^*, \quad S_n(k) = 1 \quad \text{and} \quad S_n(k) = \max_{1 \leq j \leq r} \{ S_j(i_jk) \}, \]

where \( n = p_1^{i_1} \cdot p_2^{i_2} \cdots p_r^{i_r} \) and \( S_j \) are functions defined in [4].

They \( \Sigma_1 \)-standardise \((\mathbb{N}^*,+)\) in \((\mathbb{N}^*,\leq,+)\) in the sense that

\[ \Sigma_1: \quad \max \{ S_n(a), S_n(b) \} \leq S_n(a + b) \leq S_n(a) + S_n(b) \]

for every \( a, b \in \mathbb{N}^* \) and \( \Sigma_2 \)-standardise \((\mathbb{N}^*,+)\) in \((\mathbb{N}^*,\leq,\cdot)\) by

\[ \Sigma_2: \quad \max \{ S_n(a), S_n(b) \} \leq S_n(a \cdot b) \leq S_n(a) \cdot S_n(b), \]

for every \( a, b \in \mathbb{N}^* \).

In [2] it is proved that the functions \( S_n \) are increasing and the sequence \( \{ S_p \}_{p \in \mathbb{N}^*} \) is also increasing. It is also proved that if \( p, q \) are prime numbers, then

\[ p \cdot i < q \Rightarrow S_{p^i} < S_q \quad \text{and} \quad i < q \Rightarrow S_i < S_q, \]

where \( i \in \mathbb{N}^* \).

It would be used in this paper the formula

\[ S_p(k) = p(k - i_k), \quad \text{for same} \quad i_k \quad \text{satisfying} \quad 0 \leq i_k \leq \left\lfloor \frac{k - 1}{p} \right\rfloor, \quad \text{(see [3])} \quad (1) \]

1. Proposition. Let \( p \) be a prime number and \( k_1, k_2 \in \mathbb{N}^* \). If \( k_1 < k_2 \) then \( i_{k_1} \leq i_{k_2} \), where \( i_{k_1}, i_{k_2} \) are defined by (1).

Proof. It is known that \( S_p: \mathbb{N}^* \rightarrow \mathbb{N}^* \) and \( S_p(k) = pk \) for \( k \leq p \). If \( S_p(k) = mp^a \) with \( m, a \in \mathbb{N}^*, (m, p) = 1 \), there exist \( \alpha \) consecutive numbers:

\[ n, n + 1, \ldots, n + \alpha - 1 \]

so that

\[ k \in \{ n, n + 1, \ldots, n + \alpha - 1 \} \quad \text{and} \quad S_p(n) = S_p(n + 1) = \cdots = S(n + \alpha - 1). \]
this means that \( S_p \) is stationed the \( \alpha - 1 \) steps \((k \rightarrow k + 1)\).

If \( k_1 < k_2 \) and \( S_p(k_1) = S_p(k_2) \), because \( S_p(k_1) = p(k_1 - ik_1) \), \( S_p(k_2) = p(k_2 - ik_2) \)
it results \( i_{k_1} < i_{k_2} \).

If \( k_1 < k_2 \) and \( S_p(k_1) < S_p(k_2) \), it is easy to see that we can write:

\[
  i_{k_1} = \beta_1 + \sum_a (\alpha - 1) \quad \text{where} \quad \beta_1 = 0 \quad \text{for} \quad S_p(k_1) = mp^a, \quad \text{if} \quad S_p(k_1) = mp^a
\]

then \( \beta_1 \in \{0,1,2,...,\alpha - 1\} \)

and

\[
  i_{k_2} = \beta_2 + \sum_a (\alpha - 1) \quad \text{where} \quad \beta_2 = 0 \quad \text{for} \quad S_p(k_2) = mp^a, \quad \text{if} \quad S_p(k_2) = mp^a \quad \text{then}
\]

\[
  mp^a < S_p(k_2)
\]

\( \beta_2 \in \{0,1,2,...,\alpha - 1\} \).

Now is obviously that \( k_1 < k_2 \) and \( S_p(k_1) < S_p(k_2) \) \( \Rightarrow i_{k_1} \leq i_{k_2} \). We note that, for

\( k_1 < k_2 \), \( i_{k_1} = i_{k_2} \iff S_p(k_1) < S_p(k_2) \) and \( \{mp^a|\alpha > 1 \text{ and } mp^a \leq S_p(k_1)\} = \{mp^a|\alpha > 1 \text{ and } mp^a \leq S_p(k_2)\} \)

2. Proposition. If \( p \) is a prime number and \( p > 5 \), then \( S_p > S_{p-1} \) and \( S_p > S_{p+1} \).

Proof. Because \( p - 1 < p \) it results that \( S_{p-1} < S_p \). Of course \( p + 1 \) is even and so:

(i) if \( p + 1 = 2' \), then \( i > 2 \) and because \( 2i < 2' - 1 = p \) we have \( S_{p+1} < S_p \).

(ii) if \( p + 1 = 2' \), let \( p + 1 = p_1' \cdot p_2' \cdots p_r' \), then \( S_{p+1}(k) = \max_{i \in S_p} (p_j') = S_{p_m}(k) = S_{p_m}(i_m \cdot k) \).

Because \( p_m \cdot i_m \leq p_m^a \leq \frac{p + 1}{2} < p \) it results that \( S_{p_m}(k) < S_p(k) \) for \( k \in N^* \), so that

\( S_{p+1} < S_p \).

3. Proposition. Let \( p, q \) be prime numbers and the sequences of functions

\[
  \{ S_{p_j} \}_{j \in N^*}, \quad \{ S_{q_j} \}_{j \in N^*}
\]

If \( p < q \) and \( i \leq j \), then \( S_{p_i} < S_{q_j} \).

Proof. Evidently, if \( p < q \) and \( i \leq j \), then for every \( k \in N^* \)

\[
  S_{p_i}(k) \leq S_{p_i}(k) < S_{q_j}(k)
\]

so, \( S_{p_i} < S_{q_j} \).

4. Definition. Let \( p, q \) be prime numbers. We consider a function \( S_{q_i} \), a sequence of functions \( \{ S_{q_i} \}_{i \in N^*} \), and we note:

\[
  i_{(j)} = \max \{ i | S_{q_i} < S_{p_j} \}
\]
\[ i_{(j)} = \min \left\{ i \mid S_{q_i} < S_{q_j} \right\}. \]

Then \( \{ k \in N \mid i_{(j)} < k < i_{(j')} \} = \Delta_{\rho(q')} = \Delta_{(j)} \) defines the interference zone of the function \( S_{q_i} \) with the sequence \( \left\{ S_{\rho_i} \right\}_{i \in N^*} \).

5. Remarque.
   a) If \( S_{q_i} < S_{q_j} \) for \( i \in N^* \), then now exists \( i_{(j)} \) and \( j \), and we say that \( S_{q_j} \) is separately of the sequence of functions \( \left\{ S_{\rho_i} \right\}_{i \in N^*} \).
   b) If there exist \( k \in N^* \) so that \( S_{\rho_k} < S_{q_i} < S_{\rho_{k+1}} \), then \( \Delta_{\rho(q')} = \emptyset \) and say that the function \( S_{q_i} \) does not interfere with the sequence of functions \( \left\{ S_{\rho_i} \right\}_{i \in N^*} \).

6. Definition. The sequence \( \{ x_n \}_{n \in N^*} \) is generally increasing if

\[ \forall n \in N^* \exists m_0 \in N^* \text{ so that } x_m \geq x_n \text{ for } m \geq m_0. \]

7. Remarque. If the sequence \( \{ x_n \}_{n \in N^*} \) with \( x_n \geq 0 \) is generally increasing and bounded, then every subsequence is generally increasing and bounded.

8. Proposition. The sequence \( \{ S_n(k) \}_{n \in N^*} \), where \( k \in N^* \), is in generally increasing and bounded.

   Proof. Because \( S_n(k) = S_{n_k}(1) \), it results that \( \{ S_n(k) \}_{n \in N^*} \) is a subsequence of \( \{ S_{m}(1) \}_{m \in N^*} \).

   The sequence \( \{ S_{m}(1) \}_{n \in N^*} \) is generally increasing and bounded because:

\[ \forall m \in N^* \exists t_0 = m! \text{ so that } \forall t \geq t_0 S_t(1) \geq S_{t_0}(1) = m \geq S_m(1). \]

From the remark 7 it results that the sequence \( \{ S_n(k) \}_{n \in N^*} \) is generally increasing bounded.

9. Proposition. The sequence of functions \( \{ S_n \}_{n \in N^*} \) is generally increasing bounded.

   Proof. Obviously, the zone of interference of the function \( S_n \) with \( \{ S_m \}_{m \in N^*} \) is the set

\[ \Delta_{n(m)} = \{ k \in N^* \mid n_{(m)} < k < n^{(m)} \} \text{ where} \]

\[ n_{(m)} = \max \{ n \in N^* \mid S_n < S_m \} \]

\[ n^{(m)} = \min \{ n \in N^* \mid S_m < S_n \} \text{.} \]
The interference zone $\Delta_{m(m)}$ is nonempty because $S_m \in \Delta_{m(m)}$ and finite for $S_1 \leq S_m \leq S_p$, where $p$ is one prime number greater than $m$.

Because $\{S_n(1)\}$ is generally increasing it results:

$$\forall m \in \mathbb{N}^* \exists t_0 \in \mathbb{N}^* \text{ so that } S_t(1) \geq S_m(1) \text{ for } \forall t \geq t_0.$$ 

For $t_0 = t_0 + n^{(m)}$ we have

$$S_r \geq S_m \geq S_m(1) \text{ for } \forall r \geq t_0,$$

so that $\{S_n\}_{n \in \mathbb{N}^*}$ is generally increasing bounded.

10. Remarque.

a) For $n = p_1^{i_1} \cdot p_2^{i_2} \cdots p_r^{i_r}$ are possible the following cases:

1) $\exists k \in \{1, 2, \ldots, r\}$ so that

$$S_{p_k^{i_k}} \leq S_{p_j^{i_j}} \text{ for } j \in \{1, 2, \ldots, r\},$$

then $S_n = S_{p_k^{i_k}}$ and $p_k^{i_k}$ is named the dominant factor for $n$.

2) $\exists k_1, k_2, \ldots, k_m \in \{1, 2, \ldots, r\}$ so that:

$$\forall t \in \overline{1, m} \exists q_t \in \mathbb{N}^* \text{ so that } S_n(q_t) = S_{p_{k_t}^{i_t}}(q_t) \text{ and}$$

$$\forall l \in \mathbb{N}^* \ S_n(I) = \max_{1 \leq t \leq m} \left\{ S_{p_{k_t}^{i_t}}(I) \right\}.$$ 

We shall name $\{p_{k_t}^{i_t} \mid t \in \overline{1, m}\}$ the active factors, the others would be name passive factors for $n$.

b) We consider

$$N_{p_1, p_2} = \{n = p_1^{i_1} \cdot p_2^{i_2} \mid i_1, i_2 \in \mathbb{N}^*\}, \text{ where } p_1 < p_2 \text{ are prime numbers.}$$

For $n \in N_{p_1, p_2}$ appear the following situations:

1) $i_1 \in (0, i_1^{(2)})$, this means that $p_1^{i_1}$ is a passive factor and $p_2^{i_2}$ is an active factor.

2) $i_1 \in (i_1^{(2)}, i_1^{(2)})$ this means that $p_1^{i_1}$ and $p_2^{i_2}$ are active factors.

3) $i_1 \in [i_1^{(2)}, \infty)$ this means that $p_1^{i_1}$ is a active factor and $p_2^{i_2}$ is a passive factor.
For $p_1 < p_2$ the repartition of exponents is representively in following scheme:

For numbers of type 2) $i_1 \in (i_{(p_3)}, i_{(p_2)})$ and $i_2 \in (i_{(p_2)}, i_{(p_1)})$

c) I consider that

$$N_{p_1p_2p_3} = \{n = p_1^{i_1} \cdot p_2^{i_2} \cdot p_3^{i_3} | i_1, i_2, i_3 \in \mathbb{N}^*\},$$

where $p_1 < p_2 < p_3$ are prime numbers.

Exist the following situations:

1) $n \in N^{p_j}, j = 1, 2, 3$ this means that $p_j^{i_j}$ is active factor.

2) $n \in N^{p_jp_k}, j = k; j, k \in \{1, 2, 3\}$, this means that $p_j^{i_j}, p_k^{i_k}$ are active factors.

3) $n \in N^{p_1p_2p_3}$, this means that $p_1^{i_1}, p_2^{i_2}, p_3^{i_3}$ are active factors. $N^{p_1p_2p_3}$ is named the S-active cone for $N_{p_1p_2p_3}$.

Obviously

$$N^{p_1p_2p_3} = \{n = p_1^{i_1} \cdot p_2^{i_2} \cdot p_3^{i_3} | i_1, i_2, i_3 \in \mathbb{N}^* \text{ and } i_k \in (i_{(p_k)}, i_{(p_j)}) \text{ where } j \neq k; j, k \in \{1, 2, 3\}\}.$$

The repartition of exponents is represented in the following scheme:
For $p_1 < p_2$ the repartition of exponents is represently in following scheme:

For numbers of type 2) $i_1 \in (i_{1(t_2)}, i_1^{(t_2)})$ and $i_2 \in (i_{2(t_2)}, i_2^{(t_2)})$

c) I consider that

$$N_{n_1, n_2, n_3} = \{n = p_1^{i_1} \cdot p_2^{i_2} \cdot p_3^{i_3} | i_1, i_2, i_3 \in \mathbb{N}^*\},$$

where $p_1 < p_2 < p_3$ are prime numbers.

Exist the following situations:

1) $n \in N_{n_1}$, $j = 1, 2, 3$ this means that $p_j^{i_j}$ is active factor.

2) $n \in N_{n_1, n_2}$, $j = k, j, k \in \{1, 2, 3\}$, this means that $p_j^{i_j}, p_k^{i_k}$ are active factors.

3) $n \in N_{n_1, n_2, n_3}$, this means that $p_1^{i_1}, p_2^{i_2}, p_3^{i_3}$ are active factors. $N_{n_1, n_2, n_3}$ is named the S-active cone for $N_{n_1, n_2, n_3}$.

Obviously

$$N_{n_1, n_2, n_3} = \{n = p_1^{i_1} \cdot p_2^{i_2} \cdot p_3^{i_3} | i_1, i_2, i_3 \in \mathbb{N}^* \text{ and } i_k \in (i_{1(k)}, i_k^{(t_k)}) \text{ where } j = k; j, k \in \{1, 2, 3\}\}.$$ 

The repartition of exponents is represented in the following scheme:
d) Generally, I consider \( N_{p_1 \cdots p_r} = \{ n = p_1^{i_1} \cdot p_2^{i_2} \cdots \cdot p_r^{i_r} \mid i_1, i_2, \ldots, i_r \in \mathbb{N}^+ \} \), where \( p_1 < p_2 < \cdots < p_r \) are prime numbers.

On \( N_{p_1 \cdots p_r} \) exist the following relation of equivalence:

\[ n \sim m \iff n \text{ and } m \text{ have the same active factors.} \]

This have the following classes:

- \( N_{p^n} \), where \( j_1 \in \{1, 2, \ldots, r\} \).

\( n \in N_{p^n} \iff n \) has only \( p_{j_1}^n \) active factor

- \( N_{p_{j_1} p_{j_2}^n} \), where \( j_1 \neq j_2 \) and \( j_1, j_2 \in \{1, 2, \ldots, r\} \).

\( n \in N_{p_{j_1} p_{j_2}^n} \iff n \) has only \( p_{j_1}^n \), \( p_{j_2}^n \) active factors.

\( N_{p_{j_1} \cdots p_{j_r}^n} \) which is named \( S \)-active cone.

\[ N_{p_{j_1} \cdots p_{j_r}^n} = \{ n \in N_{p_{j_1} \cdots p_{j_r}^n} \mid n \text{ has } p_{j_1}^n, p_{j_2}^n, \ldots, p_{j_r}^n \text{ active factors} \}. \]

Obviously, if \( n \in N_{p_{j_1} \cdots p_{j_r}^n} \), then \( i_k \in (i_{k'(j)}, i_{k''(j)}) \) with \( k \neq j \) and \( k, j \in \{1, 2, \ldots, r\} \).

\[ \text{REFERENCES} \]


