On the mean value of the Pseudo-Smarandache-Squarefree function

Xuhui Fan†‡ and Chengliang Tian†

† Department of Mathematics, Northwest University, Xi’an, Shaanxi, 710069
‡ Foundation Department, Engineering College of Armed Police Force, Xi’an, Shaanxi, 710086

Abstract For any positive integer \( n \), the Pseudo Smarandache Squarefree function \( Z_w(n) \) is defined as \( Z_w(n) = \min\{ m : n \mid m^n, m \in \mathbb{N} \} \), and the function \( Z(n) \) is defined as \( Z(n) = \min\{ m : n \leq \frac{m(m+1)}{2}, m \in \mathbb{N} \} \). The main purpose of this paper is using the elementary methods to study the mean value properties of the function \( Z_w(Z(n)) \), and give a sharper mean value formula for it.

Keywords Pseudo-Smarandache-Squarefree function \( Z_w(n) \), function \( Z(n) \), mean value, asymptotic formula.

§1. Introduction and result

For any positive integer \( n \), the Pseudo-Smarandache-Squarefree function \( Z_w(n) \) is defined as the smallest positive integer \( m \) such that \( n \mid m^n \). That is,

\[
Z_w(n) = \min\{ m : n \mid m^n, m \in \mathbb{N} \}.
\]

For example \( Z_w(1) = 1, Z_w(2) = 2, Z_w(3) = 3, Z_w(4) = 2, Z_w(5) = 5, Z_w(6) = 6, Z_w(7) = 7, Z_w(8) = 2, Z_w(9) = 3, Z_w(10) = 10, \cdots \). About the elementary properties of \( Z_w(n) \), some authors had studied it, and obtained some interesting results. For example, Felice Russo [1] obtained some elementary properties of \( Z_w(n) \) as follows:

Property 1. The function \( Z_w(n) \) is multiplicative. That is, if \( GCD(m, n) = 1 \), then \( Z_w(m \cdot n) = Z_w(m) \cdot Z_w(n) \).

Property 2. \( Z_w(n) = n \) if and only if \( n \) is a squarefree number.

The main purpose of this paper is using the elementary method to study the mean value properties of \( Z(Z(n)) \), and give a sharper asymptotic formula for it, where \( Z(n) \) is defined as \( Z(n) = \min\{ m : n \leq \frac{m(m+1)}{2}, m \in \mathbb{N} \} \). That is, we shall prove the following conclusion:

Theorem. For any real number \( x \geq 2 \), we have the asymptotic formula

\[
\sum_{n \leq x} Z_w(Z(n)) = \left(1 + \prod_p \left(1 + \frac{1}{p(p^2-1)}\right)\right) \cdot \frac{4\sqrt{2}}{\pi^2} \cdot x^{3/2} + O(x^{5/4}),
\]

where \( \prod_p \) denotes the product over all primes.
§2. Some lemmas

To complete the proof of the theorem, we need the following several lemmas.

**Lemma 1.** For any real number \( x \geq 2 \), we have the asymptotic formula
\[
\sum_{m \leq x} \mu^2(m) = \frac{6}{\pi^2} x + O(\sqrt{x}).
\]

**Proof.** See reference [2].

**Lemma 2.** For any real number \( x \geq 2 \), we have the asymptotic formula
\[
\sum_{m \leq x} m^2 = \frac{2}{\pi^2} x^3 + O\left(x^{\frac{5}{2}}\right),
\]
where \( A \) denotes the set of all square-free integers.

**Proof.** By the Abel’s summation formula (See Theorem 4.2 of [3]) and Lemma 1, we have
\[
\sum_{m \leq x} m^2 \mu^2(m) = x \cdot \left(\frac{6}{\pi^2} x + O(\sqrt{x})\right) - 2 \int_1^x t \left(\frac{6}{\pi^2} t + O(\sqrt{t})\right) dt
\]
\[
= \frac{6}{\pi^2} x^3 + O\left(x^{\frac{5}{2}}\right) - 4 \frac{\pi^2}{2} x^3 = \frac{2}{\pi^2} x^3 + O\left(x^{\frac{5}{2}}\right).
\]
This proves Lemma 2.

**Lemma 3.** For any real number \( x \geq 2 \) and \( s > 1 \), we have the inequality
\[
\sum_{m \leq x} Z_w(m) / m^s < \prod_p \left(1 + \frac{1}{p^s-1(p^s-1)}\right).
\]
Specially, if \( s > \frac{3}{2} \), then we have the asymptotic formula
\[
\sum_{m \leq x} Z_w(m) / m^s = \prod_p \left(1 + \frac{1}{p^s-1(p^s-1)}\right) + O\left(x^{\frac{3}{2}-s}\right),
\]
where \( B \) denotes the set of all square-full integers.

**Proof.** First we define the arithmetical function \( a(m) \) as follows:
\[
a(m) = \begin{cases} 
1 & \text{if } m \in B; \\
0 & \text{otherwise.}
\end{cases}
\]

From Property 1 and the definition of \( a(m) \) we know that the function \( Z_w(m) \) and \( a(m) \) are multiplicative. If \( s > 1 \), then by the Euler product formula (See Theorem 11.7 of [3]) we have
\[
\sum_{m \leq x} \sum_{m \in B} Z_w(m) / m^s = \prod_{p} \left(1 + \frac{p}{p^s} + \frac{p}{p^{2s}} + \cdots\right)
\]
\[
= \prod_{p} \left(1 + \frac{1}{p^{s-1}(p^s-1)}\right).
\]
Note that if \( m \in B \), then \( Z_w(m) \leq \sqrt{m} \). Hence, if \( s > \frac{3}{2} \), then we have

\[
\sum_{m \leq x \atop m \in B} \frac{Z_w(m)}{m^s} = \sum_{m=1 \atop m \in B}^\infty \frac{Z_w(m)}{m^s} - \sum_{m>\sqrt{x} \atop m \in B} \frac{Z_w(m)}{m^s} \\
= \sum_{m=1 \atop m \in B}^\infty \frac{Z_w(m)}{m^s} + O\left( \sum_{m>\sqrt{x} \atop m \in B} \frac{1}{m^s-\frac{1}{2}} \right) \\
= \prod_p \left( 1 + \frac{1}{p^{s-1}(p^s - 1)} \right) + O\left( x^{\frac{3}{2} - s}\right).
\]

This proves Lemma 3.

§3. Proof of the theorem

In this section, we shall use the elementary method to complete the proof of the theorem.

Note that if \( \frac{(m-1)m}{2} + 1 \leq n \leq \frac{m(m+1)}{2} \), then \( Z(n) = m \). That is, the equation \( Z(n) = m \) has \( m \) solutions as follows:

\[
n = \frac{(m-1)m}{2} + 1, \frac{(m-1)m}{2} + 2, \ldots, \frac{m(m+1)}{2}.
\]

Since \( n \leq x \), from the definition of \( Z(n) \) we know that if \( Z(n) = m \), then \( 1 \leq m \leq \sqrt{\frac{8x+1}{2}} - 1 \).

Note that \( Z_w(n) \leq n \), we have

\[
\sum_{n \leq x} Z_w(Z(n)) = \sum_{n \leq x} Z_w(m) = \sum_{n=1 \atop Z(n) = m}^\infty m \cdot Z_w(m) + O(x) \\
= \sum_{m \leq \sqrt{\frac{8x+1}{2}}} m \cdot Z_w(m) + O(x). \tag{2}
\]

We separate all integer \( m \) in the interval \([1, \sqrt{2x}]\) into three subsets \( A \), \( B \), and \( C \) as follows:

\( A \): the set of all square-free integers; \( B \): the set of all square-full integers; \( C \): the set of all positive integer \( m \) such that \( m \in [1, \sqrt{2x}] \setminus A \cup B \).

Note that (2), we have

\[
\sum_{n \leq x} Z_w(Z(n)) = \sum_{m \leq \sqrt{\frac{8x+1}{2}}} m \cdot Z_w(m) + \sum_{m \leq \sqrt{\frac{8x+1}{2}}} m \cdot Z_w(m) + \sum_{m \leq \sqrt{\frac{8x+1}{2}}} m \cdot Z_w(m) + O(x). \tag{3}
\]

From Property 2 and Lemma 2 we know that if \( m \in A \), then we have

\[
\sum_{m \leq \sqrt{\frac{8x+1}{2}}} m \cdot Z_w(m) = \sum_{m \leq \sqrt{\frac{8x+1}{2}}} m^2 = \frac{4\sqrt{3}}{\pi^2} x^{\frac{3}{2}} + O\left( x^{\frac{5}{4}}\right). \tag{4}
\]

It is clear that if \( m \in B \), then \( Z_w(m) \leq \sqrt{m} \). Hence

\[
\sum_{m \leq \sqrt{\frac{8x+1}{2}}} m \cdot Z_w(m) \ll \sum_{m \leq \sqrt{\frac{8x+1}{2}}} m^{\frac{3}{2}} \ll x^{\frac{5}{4}}. \tag{5}
\]
If $m \in C$, then we write $m$ as $m = q \cdot n$, where $q$ is a square-free integer and $n$ is a square-full integer. From Property 1, Property 2, Lemma 2 and Lemma 3 we have

$$\sum_{m \leq \sqrt{2}x} m \cdot Z_w(m) = \sum_{n \leq \sqrt{2}x} nZ_w(n)a(n) \sum_{q \leq \frac{\sqrt{2}x}{n}} q^2 \mu^2(q) = \sum_{n \leq \sqrt{2}x} nZ_w(n)a(n) \left( \frac{4\sqrt{2}}{\pi^2} \cdot \frac{x^{\frac{3}{2}}}{n^3} + O\left(\frac{x^{\frac{3}{2}}}{n^3}\right) \right) = \frac{4\sqrt{2}}{\pi^2} x^{\frac{3}{2}} \sum_{n \leq \sqrt{2}x} Z_w(n)a(n) + O\left(\frac{x^{\frac{3}{2}}}{n^3}\right) \sum_{n \leq \sqrt{2}x} Z_w(n)a(n) = \frac{4\sqrt{2}}{\pi^2} \prod_p \left(1 + \frac{1}{p(p^2 - 1)}\right) x^{\frac{3}{2}} + O\left(x^{\frac{3}{2}}\right). \tag{6}$$

Combining (3), (4), (5) and (6), we may immediately deduce the asymptotic formula

$$\sum_{n \leq x} Z_w(Z(n)) = \left(1 + \prod_p (1 + \frac{1}{p(p^2 - 1)})\right) \cdot \frac{4\sqrt{2}}{\pi^2} \cdot x^{\frac{3}{2}} + O\left(x^{\frac{3}{2}}\right).$$

This completes the proof of Theorem.

References


