NUMERICAL FUNCTIONS AND TRIPLETS

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We consider the functions: \( f_s, f_d, f_p, F : \mathbb{N}^* \rightarrow \mathbb{N} \), where 
\[ f_s(k) = n, \quad f_d(k) = n, \quad f_p(k) = n, \quad F(k) = n, \quad n \] 
being, respectively, the least natural number such that \( k/n! - 1, \ k/n! + 1, \ k/n! \pm 1, \ k/n! \) or \( k/n! \pm 1 \). 
This functions have the next properties:

1. Obviouvsly, from definition of this function, it results: 
\[ F(k) = \min\{S(k), f_p(k)\} = \min\{S(k), f_s(k), f_d(k)\} \]

where \( S \) is the Smarandache function (see [3]).

2. \( F(k) \leq S(k), \quad F(k) \leq f_s(k), \quad F(k) \leq f_d(k), \quad F(k) \leq f_p(k) \)

3. \( F(k) = S(k) \) if \( k \) is even, \( k \geq 4 \).

Proof. For any \( n \in \mathbb{N}, \ n \geq 2, \ n! \) is even, \( n! \pm 1 \) are odd. If \( k \) is even, 
then \( k \) cannot divide \( n! \pm 1 \). So \( F(k) = S(k) = n \geq 2 \) if \( k \) is even, \( k \geq 4 \).

4. If \( p > 3 \) is prime number, then \( F(p) \leq p - 2 \).

Proof. According to Wilson's theorem \( (p-1)! + 1 = M_p \). Because 
\( (p-2)! - 1 + (p-1)! + 1 = (p-2)!p \) results for \( p > 3 \), \( (p-2)! - 1 = M_p \) 
and so \( F(p) \leq p - 2 \).

5. \( F(m!) = F(m! \pm 1) = S(m!) = m \).

6. The equation \( F(k) = F(k + 1) \) has infinitly many solutions, because, 
according to the property 5), there is the solutions \( k = m! ; \ m \in \mathbb{N}^* \).
7. If \( F(k) = S(k) \) and \( n \) is the least natural number such that \( k/n! \), then \( k \) not divide \( s! \pm 1 \) for \( s < n \).

Let \( k = p_1^{a_1} \cdot p_2^{a_2} \cdots p_r^{a_r} \). According to \( S(k) = \max \{ S_p(\alpha_i) \} \), it results that \( S(k) \geq p_h \), where \( p_h = \min\{p_1, p_2, \ldots, p_r\} \).

If \( k \) not divide \( s! \pm 1 \) for \( s \leq p_h \), then \( k \) not divide \( t! \pm 1 \) for \( t > p_h \).

Consequently, if \( k \) not divide \( (n-1)! \), \( k/n! \) and \( k \) not divide \( s! \pm 1 \) for \( s \leq \min\{n, p_h\} \), then \( F(k) = S(k) = n \).

Obviously, the numbers \( k = 3t \), \( t \) being odd, \( t \neq 1 \), have \( p_h = 3 \) and they satisfy the condition \( 3t \) not divide \( s! \pm 1 \) for \( s = 1, 2, 3 \).

Therefore, for \( k = 3t \), \( t \) odd, \( t \neq 1 \), \( F(3t) = S(3t) = n \), \( n \) being the least natural number such that \( 3t/n! \).

8. The partition "bai" of the odd numbers.

\[ A = \{ k \in \mathbb{N} | k \text{ odd and } F(k) = S(k) \} \]
\[ B = \{ k \in \mathbb{N} | k \text{ odd and } F(k) < S(k) \} \]

\((A, B)\) is the partition "bai" of the odd numbers.

Into \( A \) there are numbers \( k = 3t \), \( t \) odd, \( t \neq 1 \). Obviously, \( A \) has infinitely many elements.

Into \( B \) there are numbers \( k = t! \pm 1 \) with \( t \geq 3 \), \( t \in \mathbb{N} \). Obviously, \( B \) has infinitely many elements.

**Definition 1** Let \( n \in \mathbb{N}^* \). We called triplet \( \hat{n} \), the set:

\( \hat{n} = n - 1, n, n + 1 \).

**Definition 2** Let \( k < n \). The triplets \( \hat{k} \), \( \hat{n} \) are separated if \( k + 1 < n - 1 \), i.e. \( n - k > 2 \).

**Definition 3** The triplets \( \hat{k} \), \( \hat{n} \) are \( l_s \)-relatively prime if \( (k - 1, n - 1) = 1 \), \( (k + 1, n + 1) \neq 1 \).

For example: \( \hat{6} \) and \( \hat{72} \) are \( l_s \)-relatively prime.

**Definition 4** The triplets \( \hat{k} \), \( \hat{n} \) are \( l_d \)-relatively prime if \( (k - 1, n - 1) \neq 1 \), \( (k + 1, n + 1) = 1 \).

**Definition 5** The triplets \( \hat{k} \), \( \hat{n} \) are \( l \)-relatively prime if \( (k - 1, n - 1) = 1 \), \( (k + 1, n + 1) = 1 \).
**Definition 6** The triplets \( \hat{k}, \hat{n} \) are \( d \)-relatively prime if
\[
(k - 1, n + 1) = 1, (k + 1, n - 1) = 1.
\]

For example: \( \hat{2} \) and \( \hat{6} \) are \( d \)-relatively prime.

**Definition 7** Let \( k < n \). The triplets \( \hat{k}, \hat{n} \) are \( d_s \)-relatively prime if
\[
(k - 1, n + 1) = 1, (k + 1, n - 1) \neq 1.
\]

For example: \( \hat{6} \) and \( \hat{120} \) are \( d_s \)-relatively prime.

**Definition 8** Let \( k < n \). The triplets \( \hat{k}, \hat{n} \) are \( d_d \)-relatively prime if
\[
(k - 1, n + 1) = 1, (k + 1, n - 1) = 1.
\]

Example: \( \hat{6} \) and \( \hat{24} \) are \( d_d \)-relatively prime.

**Definition 9** The triplets \( \hat{k}, \hat{n} \) are \( p \)-relatively prime if
\[
(k - 1, n - 1) = 1, (k - 1, n + 1) = 1, (k + 1, n - 1) = 1, (k + 1, n + 1) = 1.
\]

Obviously, if \( \hat{k}, \hat{n} \) are \( p \)-relatively prime, then they are \( l \) and \( d \)-relatively prime.

For example: \( \hat{6} \) and \( \hat{24} \) are \( p \)-relatively prime.

**Definition 10** Let \( k < n \). The triplets \( \hat{k}, \hat{n} \) are \( F \)-relatively prime if
\[
(k - 1, n - 1) = 1, (k + 1, n - 1) = 1,
(k - 1, n) = 1, (k + 1, n) = 1
(k - 1, n + 1) = 1, (k + 1, n + 1) = 1.
\]

**Definition 11** The triplets \( \hat{k}, \hat{n} \) are \( t \)-relatively prime if
\[
(k - 1, n - 1) \cdot (k - 1, n) \cdot (k - 1, n + 1) \cdot (k, n - 1) \cdot (k, n) \cdot (k, n + 1) \cdot
(k + 1, n - 1) \cdot (k + 1, n) \cdot (k + 1, n + 1) = 6.
\]

For example: \( \hat{2} \) and \( \hat{4} \) and \( t \)-relatively prime.

**Definition 12** Let \( H \subset \mathbb{N}^* \). The triplet \( \hat{n}, n \in H \) is, respectively,
\( l_s, l_d, l, d, d_s, d_d, p, F \), \( t \)-prime concerned at \( H \), if \( \forall s \in H, s < n \), the triplets \( \hat{s}, \hat{n} \) are, respectively, \( l_s, l_d, l, d, d_s, d_d, p, F \), \( t \)-relatively prime.

Let \( H = \{n! | n \in \mathbb{N}^*\} \). For the triplets \( \hat{m}, m \in H \) there are particular properties.

**Proposition 1** Let \( k < n \). The triplets \( \hat{(k!)} \), \( \hat{(n!)} \) are separated if
\( n > \max\{2, k\} \).

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Proof. Obviously, $n! - k! > 2$ if $n > 2$ and $k < n$, i.e. $n > \max\{2, k\}$.

**Proposition 2** Let $n > \max\{2, k\}$ and $M_{kn} = \{m \in \mathbb{N}|k! + 1 < m < n! - 1\}$. If $k_1 < k_2$ and $n_1 > \max\{2, k_1\}$, $n_2 > \max\{2, k_2\}$, then $n_1 - k_1 \leq n_2 - k_2 \Rightarrow \text{card} M_{k_1n_1} < \text{card} M_{k_2n_2}$.

**Proof.** For $n > k \geq 2$ it is true that

$$n! - (n - 1)! > k! - (k - 1)!$$

Let $n > k \geq 2$, $1 \leq s \leq k$. Using (1) we can write:

$$n! - (n - 1)! > k! - (k - 1)!$$

$$\left(\frac{n - 1}{n - s - 1}\right)! - \left(\frac{n - 1}{n - s}\right)! > \left(\frac{k - 1}{k - s - 1}\right)! - \left(\frac{k - 1}{k - s}\right)!$$

By summing this inequalities, it results:

$$n! - (n - s)! > k! - (k - s)!$$

Let $2 \leq k_1 < n_1$, $2 \leq k_2 < n_2$, $k_1 < k_2$, $n_1 - k_1 \leq n_2 - k_2$. Then $n_2 - n_1 \geq k_2 - k_1 \geq 1$ and there is $n_3$ such that $n_2 > n_3 \geq n_1$ and $n_2 - n_3 = k_2 - k_1$.

Using (2) we can write:

$$n_2! - n_3! > k_2! - k_1!$$

Since $n_3! \geq n_1!$ we have:

$$n_2! - n_1! > k_2! - k_1!$$

According to $\text{card} M_{k_1n_1} = n_1! - 1 - (k_1! + 1)$, $\text{card} M_{k_2n_2} = n_2! - 1 - (k_2! + 1)$, it results that:

$$\text{card} M_{k_2n_2} - \text{card} M_{k_1n_1} = n_2! - n_1! - (k_2! - k_1!)$$

That is, taking into account (3), $\text{card} M_{k_1n_1} < \text{card} M_{k_2n_2}$.

**Definition 13** Let $k < n$. The triplets $(k!)$, $(n!)$ are linked if $k! - 1 = n$ or $k! + 1 = n$.

**Proposition 3** For $k \in \mathbb{N}^*$ there is $p$ prime number, such that for any $s \geq p$ the triplets $(k!)$, $(s!)$ are not $F$-relatively prime.
Proof. Obviously, for \( k = 1 \) and \( k = 2 \), the proposition is true.
If \( n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_r^{a_r} \) divide \( k! - 1 \) or \( k! + 1 \), then \( p_j > k \geq 3 \), for 
\( j \in \{1, 2, \ldots, i\} \).

Let \( \bar{n} = p_1 \cdot p_2 \cdots p_i \) and \( p = \max\{p_j\} \).

Obviously, \( \bar{n} \geq 3 \) because \( p > k \geq 3 \), \( \bar{n}/k! - 1 \) or \( \bar{n}/k! + 1 \).

For any \( s \geq p \), \( \bar{n}/s! \) and so, the triplets \( (k!) \), \( (s!) \) are not \( F \)-relatively prime.

Remark 1 i) Let \( k < n \). If \( (k!) \), \( (n!) \) are linked, then \( n - k = k! - k + 1 \).
If \( 2 < k_1 < n_1 \), \( (k_1!) \) with \( (n_1!) \) are linked and \( k_2 < n_2 \), \( (k_2!) \) with \( (n_2!) \) are 
linked, then \( k_1 < k_2 \Rightarrow n_1 - k_1 < n_2 - k_2 \) and in view of the proposition 2, 
results \( \text{card}M_{n_1,n_1} < \text{card}M_{n_2,n_2} \).

ii) There are twin prime numbers with the triplet \( (n!) \). For example 5 with 
7 are from \( (3!) \).

Definition 14 Considering the canonical decomposition of natural numbers
\( n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_r^{a_r} \), we define \( \bar{n} = \{p_1^{a_1}, p_2^{a_2}, \ldots, p_r^{a_r}\} \),
\( \mathcal{M} = \{\bar{n}|n \in \mathbb{N}^*\} \).

Definition 15 On \( \mathcal{M} \) we consider the relation of order \( \sqsubseteq \) defined by:
\[ \{p_1^{a_1}, p_2^{a_2}, \ldots, p_r^{a_r}\} \sqsubseteq \{q_1^{\beta_1}, q_2^{\beta_2}, \ldots, q_s^{\beta_s}\} \]
if and only if \( \{p_1, p_2, \ldots, p_r\} \subseteq \{q_1, q_2, \ldots, q_s\} \) and if \( p_i = q_j \), then \( \alpha_i \leq \beta_j \).

Remark 2 For any triplet \( (n!) \), \( n \in \mathbb{N}^* \), we consider the sets:
\( A_n = \{k \in \mathbb{N}^*|k \sqsubseteq \bar{n}_n\} \), \( A_n^* = \{k \in A_n|k \not\sqsubseteq A_n for h < n\} \)
\( B_n = \{k \in \mathbb{N}^*|k \not\sqsubseteq n! - 1\} \), \( B_n^* = \{k \in B_n|k \not\sqsubseteq B_n for h < n\} \)
\( C_n = \{k \in \mathbb{N}^*|k \not\sqsubseteq n! + 1\} \), \( C_n^* = \{k \in C_n|k \not\sqsubseteq C_n for h < n\} \)
\( M_n = \{k \in \mathbb{N}^*|k \sqsubseteq \bar{n}_n! \text{ or } k \sqsubseteq n! - 1 \text{ or } k \sqsubseteq n! + 1\} \)
\( M_n^* = \{k \in M_n|k \not\sqsubseteq M_n for h < n\} \).

It is obvious that:
\( A_n^* = S^{-1}(n) \), \( B_n^* = f_s^{-1}(n) \), \( C_n^* = f_d^{-1}(n) \), \( M_n^* = F^{-1}(n) \).

If \( k \in A_n^* \), it is said that \( k \) has a factorial signature which is equivalent with the factorial signature of \( n! \) (see [1]).

Let \( k \in B_n^* \), \( k = t_1^{i_1} \cdot t_2^{i_2} \cdots t_i^{i_i} \). Then \( \{t_r\} \not\sqsubseteq \bar{n}_n! \text{ for } r = 1, i \) and for any 
\( h < n \), there are \( t_j^{i_j} \), \( 1 \leq j \leq i \), such that \( \{t_j^{i_j}\} \not\sqsubseteq h! - 1 \).

Similarly, for \( k \in C_n^* \) : \( \{t_r\} \not\sqsubseteq \bar{n}_n! \text{ for } r = 1, i \) and for any \( h < n \), there are 
\( t_j^{i_j} \), \( 1 \leq j \leq i \), such that \( \{t_j^{i_j}\} \not\sqsubseteq h! + 1 \).
References


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