ON THE PERFECT SQUARES IN SMARANDACHE CONCATENATED SQUARE SEQUENCE

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Abstract

Let $n$ be positive integer, and let $s(n)$ denote the $n$-th Smarandache concatenated square number. In this paper we prove that if $n = 2, 3, 4, 7, 8, 9, 11, 12, 14, 16, 17, 18, 20, 21, 22, or 25 (\text{mod } 27)$, then $s(n)$ is not a square.

In [1], Marimutha defined the Smarandache concatenated square sequence $\{s(n)\}_{n=1}^\infty$ as follows:

(1) $s(1) = 1, \ s(2) = 14, \ s(3) = 149, \ s(4) = 14916, \ s(5) = 1491625, ...$

Then we called $s(n)$ the $n$-th Smarandache concatenated square number. Marimutha [1] conjectured that $s(n)$ is never a perfect square. In this paper we prove the following result:

Theorem.

If $n = 2, 3, 4, 7, 8, 9, 11, 12, 14, 16, 17, 18, 20, 21, 22, or 25 (\text{mod } 27)$, then $s(n)$ is not a perfect square.

The above result implies that the density of perfect squares in Smarandache concatenated square sequence is at most $11/27$.

Proof of Theorem. We now assume that $s(n)$ is a perfect square. Then we have

(2) $s(n) = x^2$,

were $x$ is a positive integer. Notice that $10^k \equiv 1 (\text{mod } 9)$ for any positive integer $k$.

We get from (1) and (2) that

(3) $s(n) = 1^2 - 2^2 + ... - n^2 = 1/6 \cdot n(n-1)(2n+1) \equiv x^2 (\text{mod } 9)$.

It implies that
(4) \[ n(n-1)(2n-1) \equiv 6x^2 \pmod{27}. \]

If \( n \equiv 2 \pmod{27} \), then from (4) we get \( 2\times 3 \times 5 \equiv 6x^2 \pmod{27} \). It follows that

(5) \[ x^2 \equiv 5 \pmod{9}. \]

Since 5 is not a square residue mod 9, (5) is impossible. Therefore, if \( n \equiv 2 \pmod{27} \), then \( s(n) \) is not a square.

By using some similarly elementary number theory methods, we can check that the congruence (4) does not hold for the remaining cases. The theorem is proved.

Reference: