In our paper we prove that the Smarandache function $S$ does not verify the Lipschitz condition, giving an answer to a problem proposed in [2] and we investigate also the possibility that some other functions, which involve the function $S$, verify or not verify the Lipschitz condition.

**Proposition 1** The function $\{n - S(n)\}$ does not verify the Lipschitz condition, where $S(n)$ is the smallest integer $m$ such that $m!$ is divisible by $n$. ($S$ is called the Smarandache function.)

**Proof.** A function $f : M \subseteq \mathbb{R} \to \mathbb{R}$ is Lipschitz iff the following condition holds:

$$\exists K > 0, \forall x, y \in M \Rightarrow |f(x) - f(y)| \leq K |x - y|$$

(K is called a Lipschitz constant).

We have to prove that for every real $K > 0$ there exist $x, y \in \mathbb{N}^*$ such that $|f(x) - f(y)| > K |x - y|$. Let $K > 0$ be a given real number. Let $x = p > 3K + 2$ be a prime number and consider $y = p + 1$ which is a composite number, being even. Since $x = p$ is a prime number we have $S(p) = p$. Using [1] we have $\max_{n \in \mathbb{N}^*,n \neq 1} \{S(n)/n\} = \frac{2}{3}$, then $\frac{S(x)}{x} = \frac{S(x + 1)}{x + 1} \leq \frac{2}{3}$, which implies that $S(p + 1) \leq \frac{2}{3}(p + 1) < p = S(p)$. We have

$$|S(p) - S(p + 1)| = p - S(p + 1) \geq p - \frac{2}{3}(p + 1) > \frac{3K + 2 - 2}{3} = K$$

**Remark 1.** The idea of the proof is based on the following observations:
If \( p \) is a prime number, then \( S(p) = p \), thus the point \((p, S(p))\) belongs to the line of equation \( y = x \).

If \( q \) is a composite integer, \( q \neq 4 \), then \( \frac{S(q)}{q} \leq \frac{3}{2} \) which means that the point \((q, S(q))\) is under the graph of the line of equation \( y = \frac{3}{2} x \) and above the axe \( O_{x} \).

Thus, for every consecutive integer numbers \( x, y \) where \( x = p \) is a prime number and \( y = p - 1 \), the length \( AB \) can be made as great as we need, for \( x, y \) sufficiently great.

**Remark 2.** In fact we have proved that the function \( f : \mathbb{N}^* \to \mathbb{N} \) defined by \( f(n) = S(n) - S(n - 1) \) is unbounded, which implies that the Smarandache's function is not Lipschitz.

In the sequel we study the Lipschitz condition for other functions which involve the Smarandache's function.

**Proposition 2** The function \( S_1 : \mathbb{N} \setminus \{0, 1\} \to \mathbb{N} \), \( S_1(n) = \frac{1}{S(n)} \) verify the Lipschitz condition.

**Proof.** For every \( x \geq 2 \) we have \( S(x) \geq 2 \), therefore \( 0 < \frac{1}{S(x)} \leq \frac{1}{2} \). If we take \( x \neq y \) in \( \mathbb{N} \setminus \{0, 1\} \), we have

\[
\left| \frac{1}{S(x)} - \frac{1}{S(y)} \right| \leq \frac{1}{2} \leq \frac{1}{2} |x - y|.
\]
For $x = y$ we have an equality in the relation above, therefore $S_1$ is a function which verifies the Lipschitz condition with $K = \frac{1}{2}$ and more, it is a contractant function.

**Remark 3.** In [2] it is proved that $\sum_{n \geq 2} \frac{1}{n^2}$ is divergent.

**Proposition 3** The function $S_2: \mathbb{N}^* \to \mathbb{N}^*, S_2(n) = \frac{S(n)}{n}$ verifies the Lipschitz condition.

**Proof.** For every $x, y \in \mathbb{N}, 1 < x < y$ we have $x = r$ and $y = r - m$ where $m \in \mathbb{N}^*$. In [2] is proved that

$$\frac{1}{(n-1)!} \leq \frac{S(n)}{n} \leq 1, \quad (\forall m \in \mathbb{N} \setminus \{0, 1\}).$$

Using this we have

$$\frac{|S(x) - S(y)|}{x - y} = \frac{|S(r) - S(r - m)|}{r - (r - m)} \leq 1 - \frac{1}{(r - m - 1)!} < 1 \leq |x - y|$$

therefore

$$\frac{|S(x) - S(y)|}{x - y} \leq |x - y|$$

for $x$ and $y$ as above. For $x = y$ we have an equality in the relation above. It follows that $S_2$ verifies the Lipschitz condition with $K = 1$.

**Remark 4.** Using the proof of Proposition 5 proved below, it can be shown that the Lipschitz constant $K = 1$ is the best possible. Indeed, take $x = r = p - 1$, $m = 1$ and therefore $y = p$ (with the notations from the proof of Proposition 8), with $p$ a primenumber. From the proof of Proposition 5, there is a subsequence of prime numbers $\{p_n\}_{k \geq 1}$ such that $\frac{S(p_n - 1)}{p_n - 1} \to 0$. For $k \geq 1$ we have, for a Lipschitz constant $K$ of $S_2$

$$K \geq \frac{|S(p_n - 1) - S(p_n) - 1|}{S(p_n - 1) - 1} = 1 - \frac{S(p_n - 1)}{p_n - 1} \to 1$$

Thus, $K \geq 1$.
Proposition 4 The function \( S_1 = N \setminus \{0, 1\} \rightarrow N \) defined by \( S_1(n) = \sum_{\substack{p \mid n \atop p \leq \sqrt{n}}} p \) does not verify the Lipschitz condition.

Proof. (Compare with the proof of Proposition 3.)

We have to prove that for every real \( K > 0 \) there exists \( x, y \in \mathbb{N}^* \) such that \( S_1(x) - S_1(y) > K \cdot |x - y| \).

Let \( K > 0 \) be a given real number, \( x = p \) be a prime number and \( y = x - 1 \). Using the Proposition 3 proved below, which asserts that the sequence \( \left\{ \frac{p^n - 1}{S(p^n - 1)} \right\} \) is unbounded (where \( \{p\}_{n \geq 1} \) is the prime numbers sequence), we have, for a prime number \( p \) such that \( \frac{p^n - 1}{S(p^n - 1)} > K + 1 \):

\[
\frac{x}{S(x)} - \frac{y}{S(y)} = \frac{p}{S(p)} - \frac{p - 1}{S(p - 1)} = \frac{p - 1}{S(p - 1)} - 1 > K + 1 - 1 = K = K \cdot |x - y|
\]

\( \blacksquare \)

Proposition 5 If \( \{p_n\}_{n \geq 1} \) is the prime numbers sequence, then the sequence \( \left\{ \frac{p_n - 1}{S(p_n - 1)} \right\}_{n \geq 2} \) is unbounded.

Proof. Denote \( q_n = p_n - 1 \) and let \( \tau_n \) be the number of the distinct prime numbers which appear in the prime factor decomposition of \( q_n \) for \( n \geq 2 \). We show below that \( \{\tau_n\}_{n \geq 2} \) is an unbounded sequence.

For a fixed \( k \in \mathbb{N}^* \), consider \( \pi_k \overset{\text{def}}{=} p_1 \cdots p_k \) and the arithmetic progression \( \{1 + \pi_k \cdot m\}_{m \geq 1} \). From the Dirichlet Theorem [3, pg.194], it follows that this sequence contains a subsequence \( \{1 + \pi_k \cdot m \}_{m \geq 1} \) of prime numbers: \( \pi_k \cdot m \mid 1 + \pi_k \cdot m \), therefore \( \pi_k \cdot m \mid \pi_k \cdot m - 1 = q_n \), which implies that \( \tau_n \geq k \). It shows that the sequence \( \{\tau_n\}_{n \geq 2} \) is an unbounded sequence.

If \( q_n = \prod_{i=1}^{n} p_{\beta_i}^\alpha_i \) then it is known (see [4]) that:

\[
S(q_n) = \max_{1 \leq i \leq n} \left( S\left( p_{\beta_i}^{\alpha_i} \right) \right) = S\left( p_{\beta_1}^{\alpha_1} \right) \leq q_n \cdot \beta_1
\]

thus

\[
\frac{q_n}{S(q_n)} = \frac{\prod_{i=1}^{n} p_{\beta_i}^{\alpha_i}}{S\left( p_{\beta_i}^{\alpha_i} \right)} \geq \left( \prod_{i=1}^{n} p_{\beta_i}^{\alpha_i} \right) \frac{p_{\beta_1}^{\alpha_1}}{\alpha_1} \cdot \tag{1}
\]

62
We have:

\[ u_j = \frac{a_j^{x_j - 1}}{x_j} \geq 2 \quad (2) \]

Indeed, if \( x_j = 1 \), then \( u_j = 1 \). If \( x_j > 1 \), then

\[ u_j \geq \frac{(x_j - 1)(x_j - 1)}{x_j} \geq \frac{x_j - 1}{2} \geq \frac{1}{2} \]

But \( v_n = \prod_{i=1}^{k} p_{2i}^{2i} \) has \( r_n - 1 \) prime factors and \( (r_n)_{n \geq 2} \) is unbounded, then it follows that \( (v_n)_{n \geq 2} \) is unbounded. Using this, (1) and (2), it follows that the sequence \( \left\{ \frac{r_n}{x(r_n)} \right\}_{n \geq 2} \) is unbounded.

**Remark 5.** Using the same idea, the Proposition 5 is true in a more general form:

For \( n \in \mathbb{Z} \), the sequence \( \left\{ \frac{r_n - a}{r(r_n - a)} \right\}_{r_n - a \geq 2} \) is unbounded, where \( (r_n)_{n \geq 2} \) is the prime numbers sequence.

**References**


