PROOF OF FUNCTIONAL SMARANDACHE ITERATIONS

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ABSTRACT: The paper makes use of method of Mathematics Analytic to prove Functional Smarandache Iterations of three kinds.

I. Proving Functional Smarandache Iterations of First Kind.

Let \( f: A \to A \) be a function, such that \( f(x) \leq x \) for all \( x \), and \( \min \{ f(x), x \in A \} \geq m_0 \), different from negative infinity.

Let \( f \) have \( p \geq 1 \) fix points: \( m_0 \leq x_1 < x_2 < \ldots < x_p \). [The point \( x \) is called fix, if \( f(x) = x \).]

Then:

\[ S_{fI}(x) = \text{the smallest number of iterations} \ k \text{ such that} \]

\[ f(f(\cdots f(x)\cdots)) = \text{constant.} \]

Proof: I. When \( A \subseteq \mathbb{Q} \) or \( A \subseteq \mathbb{R} \), conclusion is false.

Counterexample: Let \( A = [0, 1] \) with \( f(x) = x^2 \), then \( f(x) \leq x \), and \( x_1 = 0 \), \( x_2 = 1 \) are fix points.

Denote: \( A_n(x) = f(f(\cdots f(x)\cdots)) \), \( A_1(x) = f(x) \), \( (n=1, 2, \ldots) \).

Then \( A_n(x) = x^{2^n} \) ( \( n=1, 2, \ldots \) ).

For any fixed \( x \neq 0, x \neq 1 \), assumed that the smallest positive integer \( k \) exist, such that \( A_n(x) = a \) (constant), hence, \( A_{k+1}(x) = f(A_k(x)) = f(a) = a \), that is to say a be fix point.

So \( x^{2^{n+k}} = 0 \) or \( 1 \), \( \Rightarrow \ x = 0 \) or \( 1 \), this appear contradiction. If \( A \subseteq \mathbb{Z} \), let \( A \) be set of all rational number on \( [0, 1] \) with \( f(x) = x^2 \), using the same methods we can also deduce contradictory result.

This shows the conclusion is false where \( A \subseteq \mathbb{Q} \) or \( A \subseteq \mathbb{R} \).

II. When \( A \subseteq \mathbb{Z} \), the conclusion is true.

(1). If \( x = x_i \) ( \( x_i \) is fix point, \( i=1, \ldots, p \) ). Then \( f(x) = f(x_i) = x_i = A_1(x) \). So for any positive integer \( n \), \( A_n(x) = x_i \) ( \( i=1, \ldots, p \) ), \( \Rightarrow S_{fI}(x) = 1 \).

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(2). Let \( x \neq x_i \) (\( x \) is fixed, \( i = 1, \ldots, p \)), if \( f(x) = x_i \) (\( i = 1, \ldots, p \)), then \( S\overline{I}(x) = 1 \), if \( f(x) \neq x_i \) but \( f(f(x)) = A_2(x) = x_i \) (\( i = 1, \ldots, p \)), then \( S\overline{I}(x) = 2 \). In general, for fixed positive integer \( k \), if \( A_1(x) \neq x_i \), \( A_2(x) \neq x_i \) \( \ldots \) \( A_{k-1}(x) \neq x_i \), but \( A_k(x) = x_i \), then \( S\overline{I}(x) = k \).

(3). Let \( x \neq x_i \) (\( x \) is fixed), and for \( \forall n \in \mathbb{N} \) \( A_n(x) \neq x_i \) (\( i = 1, \ldots, p \)), this case is no exist.

Because \( x \) is fix point, \( m_0 < \ldots < A_n(x) < \ldots < A_2(x) < A_1(x) < x \). So sequence \( \{ A_n(x) \} \) is descending and exist boundary, this makes know that \( \{ A_n(x) \} \) is convergent. But, each item of \( \{ A_n(x) \} \) is integer, it is not convergent, this appear contradiction. This shows that the case is no exist.

(4). Let \( x \neq x_i \) (\( x \) is fixed, \( i = 1, \ldots, p \)), if exist the smallest positive integer \( k \) such that \( A_k(x) = a \) (\( a \neq x_i \)), it is yet unable. Because \( A_{k+1}(x) = A_k(x) = a \), \( A_{k+1}(x) = f(A_k(x)) = f(a) = a \), this shows that \( a \) is fix point, namely, \( a = x_i \), this also appear contradiction.

Combining (1), (2), (3) and (4) we have

\[ S\overline{I}(x) = \text{the smallest number of iterations } k \text{ such that } \]
\[ f(f(\cdots f(x)\cdots)) = x_i \quad (x_i \text{ is fix point, } i = 1, \ldots, p). \]

This proves Kind 1.

We easily give a simple deduction.

Let \( f : A \to A \) be a function, such that \( f(x) \leq x \) for all \( x \), and \( \min \{ f(x), x \in A \} \geq m_0 \), different from negative infinity.

Let \( f(m_0) = m_0 \), namely, \( m_0 \) is fix point, and only one.

Then: \( S\overline{I}(x) = \text{the smallest number of iterations } k \) such that

\[ f(f(\cdots f(x)\cdots)) = m_0. \]

This proves Kind 2.

2. Proving Functional Smarandache Iterations of Second Kind.

Kind 2.

Let \( g : A \to A \) be a function, such that \( g(x) > x \) for all \( x \), and let \( b > x \).

Then:

\[ S\overline{I}2(x, b) = \text{the smallest number of iterations } k \text{ such that } \]
\[ g(g(\cdots g(x)\cdots)) \geq b. \]
Proof: Firstly, denote: \[ B_n(x) = \underbrace{g(g(\ldots g(x))\ldots)}_{\text{n times}}, \quad (n=1,2,\ldots). \]

I. Let \( A \subseteq \mathbb{Z} \), for \( \forall x < b, \ x \in \mathbb{Z} \), assumed that there are not the smallest positive integer \( k \) such that \( B_k(x) \geq b \), then for \( \forall n \in \mathbb{N} \) have \( B_n(x) < b \), so \( x < B_1(x) < B_2(x) < \ldots < B_n(x) < \ldots < b \).

This makes know that \( \{ B_n(x) \} \) is convergent, but it is not convergent. This appear contradiction, then, there are the smallest \( k \) such that \( B_n(x) \geq b \).

II. Let \( A \subseteq \mathbb{Q} \) or \( A \subseteq \mathbb{R} \).

(1). For fixed \( x < b \). If \( g(x) \geq g(b) > b \), then \( B_n(x) \geq g(x) > b \) \( (n \in \mathbb{N}), \; SI2(x,b) = 1 \), if \( g(x) < g(b) \) but \( B_2(x) \geq g(b) > b \), then \( B_n(x) \geq g(b) > b \) \( (n \geq 2) \), \( SI2(x,b) = 2 \). In general, if \( B_1(x) < g(b) \), \( B_2(x) < g(b) \), \ldots \( B_k(x) < g(b) \), but \( B_k(x) \geq g(b) > b \), then \( SI2(x,b) = k \).

(2). For fixed \( x < b \), \( B_n(x) < g(b) \), \( (n \in \mathbb{N}) \) then \( x < B_1(x) < B_2(x) < \ldots < B_n(x) < \ldots < g(b) \), so \( \{ B_n(x) \} \) is convergent. Let \( \lim_{n \to \infty} B_n(x) = b^* \). \( \quad \therefore \quad B_n(x) < g(b) \) \( (n \in \mathbb{N}), \; b^* \leq g(b) \).

1). \( b^* = g(b) \). \( \quad \therefore \quad \lim_{n \to \infty} B_n(x) = b^* \). \( \quad \therefore \quad \exists \varepsilon = g(b) - b > 0, \exists \) positive integer \( k \), when \( n > k \) such that \( |B_n(x) - g(b)| < \varepsilon \). So \( B_n(x) > g(b) \). \( \quad \varepsilon = g(b) - (g(b) - b) = b. \) That is to say there are the smallest \( k \) such that \( B_n(x) > b \). 2). \( b^* < g(b) \). \( \quad \therefore \quad g(b^*) > b^* \). \( \therefore \quad \{ B_n(x) \} \) does not converge at \( g(b^*) \). \( \exists \varepsilon_0 = 0 \), \( \forall n \), \( \exists n_1 \), when \( n_1 > N \), such that \( |B_n(x) - g(b^*)| < \varepsilon_0 \), then, \( B_{n_1}(x) > g(b^*) + \varepsilon_0 \). \( \therefore \quad B_{n_1}(x) > b^* + \varepsilon_0 \). On the other hand, \( B_n(x) \leq b^* \) \( (n \in \mathbb{N}), \; \therefore \quad B_{n_1}(x) \leq b^* \) then \( b^* + \varepsilon_0 < B_{n_1}(x) \leq b^* \), but this is unable. This makes know that there is not the case.

By (1) and (2) we can deduce the conclusion is true in the case of \( A \) belong to \( \mathbb{Q} \) or \( \mathbb{R} \).

Combining I. and II., we have: for any fixed \( x > b \) there is \( SI2(x,b) = \) the smallest number of iterations \( k \) such that \( g \underbrace{g(g(\ldots g(x))\ldots)}_{\text{k times}} \geq b \).

This proves Kind 2.


Kind 3. Let \( h: A \to A \) be a function, such that \( h(x) < x \) for all \( x \), and let \( b < x \).

Then:
$SI3(x,b) = \text{the smallest number of iterations } k \text{ such that }$

$$h(h(\cdots h(x) \cdots)) \leq b.$$  

Using similar methods of proving Kind 2 we also can prove Kind 3, we well not prove again in the place.

We complete the proofs of Functional Smarandache Iterations of all kinds in the place.

REFERENCES

1. “Functional Iterations” at http://www.gallup.unm.edu/~smarandache/bases.txt
2. East China Normal University Department of Mathematics Writing, Mathematics Analytic, People’s Education Press, Shanghai, 1982-4.