In this paper are studied some properties of the numerical function \( F_s(x) : \mathbb{N} - \{0,1\} \rightarrow \mathbb{N} \) \( F_s(x) = \sum_{0 < p \leq x} S_p(x) \), where \( S_p(x) = S(p^a) \) is the Smarandache function defined in [4].

Numerical example: \( F_3(5) = S(2^2) + S(3^1) + S(5^1) \); \( F_3(6) = S(2^2) + S(3^1) + S(5^1) \).

It is known that: \( (p - 1)r + 1 \leq S(p') \leq pr \) so \( (p - 1)r < S(p') \leq pr \).

Then \( x(p_1 + p_2 + \cdots + p_{\pi(x)}) - \pi(x) < F_s(x) \leq x(p_1 + p_2 + \cdots + p_{\pi(x)}) \)

(1)

Where \( \pi(x) \) is the number of prime numbers smaller or equal with \( x \).

**PROPOSITION 1:** The sequence \( T(x) = 1 \log F_s(x) + \sum_{i=2}^{x} \frac{1}{F_s(i)} \) has limit \(-\infty\).

Proof. The inequality \( F_s(x) > x(p_1 + \cdots + p_{\pi(x)}) - \pi(x) \) implies \( -\log F_s(x) < -\log x(p_1 + \cdots + p_{\pi(x)}) - \pi(x) < -\log x(\pi(x)p_1 - \pi(x)) = -\log x - \log \pi(x) - \log(p_1 - 1) \).

Then for \( x = i \) the inequality (1) become:

\[
\frac{1}{F_s(i)} < \frac{1}{i(p_1 + \cdots + p_{\pi(i)})} < \frac{1}{i(p_1\pi(i) - \pi(i))} = \frac{1}{ip(i)(p_1 - 1)}
\]

Than \( T(x) < 1 - \log(x) - \log \pi(x) - \log(p_1 - 1) + \sum_{i=2}^{x} \frac{1}{ip(i)(p_1 - 1)} \)

\( p_1 = 2 \) \( \Rightarrow \) \( T(x) = 1 - \log x - \log \pi(x) + \sum_{i=2}^{x} \frac{1}{ip(i)} \)

\( \Rightarrow \lim_{x \to \infty} T(x) \leq 1 - \lim_{x \to \infty} \log x - \lim_{x \to \infty} \log \pi(x) + \lim_{x \to \infty} \sum_{i=2}^{x} \frac{1}{ip(i)} = 1 - \infty - \infty + L = -\infty. \)

**PROPOSITION 2:** The equation \( F_s(x) = F_s(x + 1) \) has no solution for \( x \in \mathbb{N} - \{0,1\} \).
Proof. First we consider that $x+1$ is a prime number with $x > 2$. In the particular case $x = 2$ we obtain $F_3(2) = S(2^2) = 4$, $F_3(3) = S(2^2) + S(3^2) = 4 + 9 = 13$. So $F_3(2) < F_3(3)$.

Next we shall write the inequalities:

\[
x(p_1 + \cdots + p_{\pi(x)} - \pi(x)) < F_3(x) \leq x(p_1 + \cdots + p_{\pi(x)})
\]

\[
(x + 1)(p_1 + \cdots + p_{\pi(x)} + p_{\pi(x+1)} - \pi(x+1)) < F_3(x + 1) \leq (x + 1)(p_1 + \cdots + p_{\pi(x)} + p_{\pi(x+1)})
\]

Using the reductio ad absurdum method we suppose that the equation $F_3(x) = F_3(x + 1)$ has solution. From (2) results the inequalities

\[
(x + 1)(p_1 + \cdots + p_{\pi(x)} + p_{\pi(x+1)} - \pi(x+1)) < F_3(x + 1) \leq x(p_1 + \cdots + p_{\pi(x)})
\]

From (3) results that:

\[
x(p_1 + \cdots + p_{\pi(x)}) - (x + 1)(p_1 + \cdots + p_{\pi(x)} + p_{\pi(x+1)} - \pi(x+1)) > 0
\]

\[
x(p_1 + \cdots + p_{\pi(x)}) - x(p_1 + \cdots + p_{\pi(x)}) - xp_{\pi(x+1)} + x\pi(x+1) - p_1 - \cdots - p_{\pi(x)} - p_{\pi(x+1)} +
\]

\[
\pi(x + 1) > 0.
\]

But $p_{\pi(x+1)} > \pi(x+1)$ so the difference from above is negative for $x > 0$, and we obtained a contradiction. So $F_3(x) = F_3(x + 1)$ has no solution for $x + 1$ a prime number.

Next, we demonstrate that the equation $F_3(x) = F_3(x + 1)$ has no solution for $x$ and $x + 1$ both composite numbers.

Let $p$ be a prime number satisfying conditions $p > \frac{x}{2}$ and $p \leq x - 1$. Such $p$ exists according to Bertrand's postulate for every $x \in \mathbb{N} - \{0, 1\}$. Than in the factorial of the number $p(x-1)$, the number $p$ appears at least $x$ times.

So, we have $S(p^x) \leq p(x-1)$.

But $p(x-1) < px + p-x$ (if $p > \frac{x}{2}$) and $px + p-x = (p-1)(x+1) + 1 \leq S(p^{x+1})$.

Therefore $\exists p \leq x-1$ so that $S(p^x) < S(p^{x+1})$.

Then $F_3(x) = S(p_1^x) + \cdots + S(p_2^x) + \cdots + S(p_{\pi(x)}^x)$

\[
F_3(x + 1) = S(p_1^{x+1}) + \cdots + S(p_2^{x+1}) + \cdots + S(p_{\pi(x)}^{x+1}) > F_3(x)
\]

In conclusion $F_3(x + 1) > F_3(x)$ for $x$ and $x + 1$ composite numbers. If $x$ is a prime number $\pi(x) = \pi(x+1)$ and the fact that the equation $F_3(x) = F_3(x + 1)$ has no solution has the same demonstration as above.

Finally the equation $F_3(x) = F_3(x + 1)$ has no solution for any $x \in \mathbb{N} - \{0, 1\}$.

**PROPOSITION 3.** The function $F_3(x)$ is strictly increasing function on its domain of definition.

The proof of this property is justified by the proposition 2.

**PROPOSITION 4.** $F_3(x + y) > F_3(x) + F_3(y)$ $\forall x, y \in \mathbb{N} - \{0, 1\}$.

**Proof.** Let $x, y \in \mathbb{N} - \{0, 1\}$ and we suppose $x < y$. According to the definition of $F_3(x)$ we have:
\[
F(x + y) = S(p_1^{x+y}) + \ldots + S(p_{\pi(x)}^{x+y}) + S(p_{x+y}^{x+y}) + \ldots + S(p_{\pi(x+y)}^{x+y})
\]

(4)

\[
F(x) + F(y) = S(p_1^x) + \ldots + S(p_{\pi(x)+1}^x) + S(p_1^y) + \ldots + S(p_{\pi(y)+1}^y) + S(p_{x+y}^{x+y}) + \ldots + S(p_{\pi(x+y)}^{x+y})
\]

But from (1) we have the following inequalities:

\[
A = (x + y)(p_1 + \ldots + p_{\pi(x)} + p_{\pi(x)+1} + \ldots + p_{\pi(x+y)} - \pi(x + y)) < F(x + y) \leq (x + y)(p_1 + \ldots + p_{\pi(x)} + p_{\pi(x)+1} + \ldots + p_{\pi(x+y)})
\]

(5)

and

\[
x(p_1 + \ldots + p_{\pi(x)} - \pi(x)) + y(p_1 + \ldots + p_{\pi(x)} + \ldots + p_{\pi(y)} - \pi(y)) < F(x) + F(y) \leq x(p_1 + \ldots + p_{\pi(x)}) + y(p_1 + \ldots + p_{\pi(x)} + \ldots + p_{\pi(x+y)}) = B
\]

(6)

We proof that \( B < A \).

\[
B < A \iff x(p_1 + \ldots + p_{\pi(x)}) + y(p_1 + \ldots + p_{\pi(x)} + \ldots + p_{\pi(y)}) < x(p_1 + \ldots + p_{\pi(x)}) + y(p_1 + \ldots + p_{\pi(x)} + \ldots + p_{\pi(x+y)}) - x\pi(x + y) + \pi(x + y) = x(p_1 + \ldots + p_{\pi(x+y)} - \pi(x + y)) + y(p_{\pi(y)+1} + \ldots + p_{\pi(x+y)} - \pi(x + y)) \geq 0
\]

But \( p_{\pi(x+y)} \geq \pi(x + y) \) so that the inequality from above is true.

**CONSEQUENCE:** \( F_S(xy) > F_S(x) + F_S(y) \quad \forall x, y \in N \setminus \{0,1\} \)

Because \( x \) and \( y \in N \setminus \{0,1\} \) and \( xy > x + y \) than \( F_S(xy) > F_S(x + y) > F_S(x) + F_S(y) \)

**PROPOSITION 5.** We try to find \( \lim_{n \to \infty} \frac{F_S(n)}{n^\alpha} \)

We have \( F_S(n) = \sum_{0 < p, s_n, p_n \text{ prime}} S(p_1^n) \) and:

\[
\frac{p_1 + p_2 + \ldots + p_{\pi(n)} - \pi(n)}{n^{\alpha-1}} < \frac{F_S(n)}{n^\alpha} \leq \frac{p_1 + p_2 + \ldots + p_{\pi(n)}}{n^\alpha}
\]

If \( \alpha < 1 \) than

\[
\lim_{n \to \infty} n^{1-\alpha}(p_1 + \ldots + p_{\pi(n)} - \pi(n)) = \infty \cdot \infty = +\infty \implies \lim_{n \to \infty} \frac{F_S(n)}{n^{\alpha-1}} = +\infty.
\]

If \( \alpha = 1 \) than

\[
\lim_{n \to \infty} n^{1-\alpha}(p_1 + \ldots + p_{\pi(n)} - \pi(n)) = \lim_{n \to \infty} (p_1 + \ldots + p_{\pi(n)} - \pi(n)) = +\infty \implies \lim_{n \to \infty} \frac{F_S(n)}{n^{\alpha-1}} = +\infty
\]
We consider now \( \alpha > 1 \).

We try to find \( \lim_{n \to \infty} \frac{\sum_{i=1}^{\pi(n)} p_i - \pi(n)}{n^{\alpha - 1}} \) and \( \lim_{n \to \infty} \frac{\sum_{i=1}^{\pi(n)} p_i}{n^{\alpha - 1}} \) applying Stolz - Cesaro:

Let \( a_n = \sum_{i=1}^{\pi(n)} p_i - \pi(n) \) and \( b_n = n^{\alpha - 1} \).

\[
\frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \frac{\sum_{i=1}^{\pi(n+1)} p_i - \pi(n+1) - \sum_{i=1}^{\pi(n)} p_i + \pi(n)}{(n+1)^{\alpha - 1} - n^{\alpha - 1}} = \begin{cases} 
\frac{n}{(n+1)^{\alpha - 1} - n^{\alpha - 1}} & \text{if } (n+1) \text{ is a prime} \\
0, & \text{otherwise}
\end{cases}
\]

Let \( c_n = \sum_{i=1}^{\pi(n)} p_i \) and \( d_n = n^{\alpha - 1} \).

\[
\frac{c_{n+1} - c_n}{d_{n+1} - d_n} = \frac{\sum_{i=1}^{\pi(n+1)} p_i - \sum_{i=1}^{\pi(n)} p_i}{(n+1)^{\alpha - 1} - n^{\alpha - 1}} = \frac{p_{\pi(n+1)}}{(n+1)^{\alpha - 1} - n^{\alpha - 1}} = \begin{cases} 
\frac{n+1}{(n+1)^{\alpha - 1} - n^{\alpha - 1}} & \text{if } (n+1) \text{ is a prime} \\
0, & \text{otherwise}
\end{cases}
\]

First we consider the limit of the function.

\[
\lim_{x \to \infty} \frac{x}{(x+1)^{\alpha - 1} - x^{\alpha - 1}} = \lim_{x \to \infty} \frac{1}{(x+1)^{\alpha - 1} - x^{\alpha - 2}} = 0 \quad \text{for } \alpha - 2 > 1
\]

We used the l'Hospital theorem:

In the same way we have

\[
\lim_{x \to \infty} \frac{x+1}{(x+1)^{\alpha - 1} - x^{\alpha - 1}} = 0 \quad \text{for } \alpha > 3.
\]

So, for \( \alpha > 3 \) we have:

\[
\lim_{x \to \infty} \frac{p_1 + p_2 + \cdots + p_{\pi(n)} - \pi(n)}{n^{\alpha - 1}} = 0 \quad \text{and}
\]

\[
\lim_{x \to \infty} \frac{p_1 + p_2 + \cdots + p_{\pi(n)}}{n^{\alpha - 1}} = 0. \quad \text{So } \lim_{x \to \infty} \frac{F(n)}{n^\alpha} = 0.
\]

Finally \( \lim_{x \to \infty} \frac{F(n)}{n^\alpha} = \begin{cases} 
0 & \text{for } \alpha > 3 \\
+\infty & \text{for } \alpha \leq 1
\end{cases}\)
BIBLIOGRAPHY


A linear combination with Smarandache Function to obtain the Identity, Proceedings of 26th Annual Iranian Mathematic Conference Shahid Baharar University of Kerman, Kerman - Iran March 28 - 31 1995