SOME PROPERTIES OF SMARANDACHE FUNCTIONS OF THE TYPE I

by

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We consider the construction of Smarandache functions of the type $I S_p^{(p|\mathbb{N}^*, p \text{ prim})}$ which are defined in [1] and [2] as follows:

$$S_n : \mathbb{N}^* \rightarrow \mathbb{N}^* : S(k) = 1 ; S(k) = \max \{ S(i, k) \} \quad \text{for } n = p_1^i p_2^j \ldots p_r^r$$

In this paper there are presented some properties of these functions. We shall study the monotonicity of each function $S_n$ and also the monotonicity of some subsequences of the sequence $(S_n)_n \in \mathbb{N}^*$.

1. Proposition. The function $S_n$ is monotonous increasing for every positive integer $n$.

Proof. The function $S_1$ is obviously monotonous increasing.

Let $k_1 < k_2$ where $k_1, k_2 \in \mathbb{N}^*$. Supposing that $n$ is a prime number and taking account that $(S(k_2)_n)! = \text{multiple} \quad n_1^i = \text{multiple} \quad n_2^j$, we have
It results that $S_n(k) \leq S_n(k)$, therefore $S_n$ is monotonous increasing. Let

$$S_n(k) = \max_{i_1 \leq i_2 \leq k} (S(i_1, k)) = S(i_1, k)$$

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Because

$$S_{m_1 m_2}(i_1, k) \leq S_{i_1 m_2}(i_1, k) \leq S_{i_1 i_2}(i_1, k)$$

it results that $S_n(k) \leq S_n(k)$ so $S_n$ is monotonous increasing.

2. Proposition. The sequence of functions $(S_p)_{p \in \mathbb{N}}$ is monotonous increasing, for every prime number $p$.

Proof. For any two numbers $i_1, i_2 \in \mathbb{N}^*$, $i_1 < i_2$ and for any $n \in \mathbb{N}^*$ we have:

$$S(p_1, n) = S(p_1, n) \leq S(p_2, n) = S(p_2, n)$$

Therefore $S(p_1) \leq S(p_2)$.

Hence the sequence $(S_p)_{p \in \mathbb{N}^*}$ is monotonous increasing for every prime number $p$.

3. Proposition. Let $p$ and $q$ two given prime numbers. If $p < q$ then

$$S_p(k) < S_q(k), \quad k \in \mathbb{N}^*$$

Proof. Let the sequence of coefficients (see [2]) $a_1^{(p)}, a_2^{(p)}, \ldots, a_n^{(p)}, \ldots$

Every $k \in \mathbb{N}^*$ can be uniquely written as

$$k = t_1 a_1^{(p)} + t_2 a_2^{(p)} + \ldots + t_n a_n^{(p)}$$

(1)
where $0 \leq t_i \leq p-1$, for $i = 1, \ldots, s-1$, and $0 \leq t_s \leq p$.

The procedure of passing from $k$ to $k+1$ in formula (1) is:

1. $t_s$ is increasing with a unity.
2. If $t_e$ cannot increase with a unity, then $t_{e-1}$ is increasing with a unity and $t_e = 0$.
3. If neither $t_e$ nor $t_{e-1}$ are not increasing with a unity then $t_{e-2}$ is increasing with a unity and $t_e = t_{e-1} = 0$.

The procedure is continued in the same way until we obtain the expression of $k+1$.

Denoting $A_k(S_p) = S_p(k+1) - S_p(k)$ the leap of the function $S_p$ when we pass from $k$ to $k+1$ corresponding to the procedure described above. We find that:

- in the case (i) $\Delta_k(S_p) = p$
- in the case (ii) $\Delta_k(S_p) = 0$
- in the case (iii) $\Delta_k(S_p) = 0$

It is obviously seen that: $S_p(n) = \sum_{k=1}^{n} A_k(n) + S_p(1)$.

Analogously we write $S_q(n) = \sum_{k=1}^{n} A_q(n) + S_q(1)$.

Taking into account that $S_p(1) = p < q = S_q(1)$ and using the procedure of passing from $k$ to $k+1$ we deduce that the number of leaps with zero value of $S_p$ is greater than the number of leaps with zero value of $S_q$, respectively the number of leaps with value $p$ of $S_p$ is less than the number of leaps of $S_q$ with value
q it result that
\[ \sum_{k=1}^{n} \frac{A_k(S)}{k} + S(1) < \sum_{k=1}^{n} \frac{A_k(S)}{q} + S(1) \quad (2) \]

Hence \( S(n) < S(n) \quad , n \in \mathbb{N}^* \).

As an example we give a table with \( S_2 \) and \( S_3 \) for \( 0 < n < 21 \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_2(k) )</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>16</td>
<td>18</td>
<td>18</td>
<td>18</td>
<td>18</td>
<td>20</td>
<td>22</td>
<td>24</td>
<td></td>
</tr>
<tr>
<td>( S_3(k) )</td>
<td>3</td>
<td>6</td>
<td>9</td>
<td>9</td>
<td>12</td>
<td>15</td>
<td>18</td>
<td>18</td>
<td>21</td>
<td>24</td>
<td>27</td>
<td>27</td>
<td>30</td>
<td>33</td>
<td>36</td>
<td>38</td>
<td>39</td>
<td>42</td>
<td>45</td>
<td></td>
</tr>
</tbody>
</table>

Hence \( S_2(k) < S_3(k) \) for \( k = 1, 2, \ldots, 20 \).

4. Remark. For any monotonous increasing sequence of prime numbers

\[ p_1 < p_2 < \ldots < p_n < \ldots \] it results that

\[ S_1 < S_{p_1} < S_{p_2} < \ldots < S_{p_n} < \ldots \]

If \( n = p_1^i p_2^i \ldots p_t^i \) and \( p_1 < p_2 < \ldots < p_t \) then

\[ S_n(k) = \max (S_{p_1^i}(k) , \ldots , S_{p_t^i}(k)) = S_{p_1^i}(k) = S_{p_i}(k) \]

5. Proposition. If \( p \) and \( q \) are prime numbers and \( p < q \) then \( S_{p_i} < S_{q_i} \).

Proof. Because \( p < q \) it results

\[ S_{p_i}(1) = p < q = S_{q_i}(1) \quad (3) \]

and \( S_{p_i}(k) = S_{p_i}(i) \leq p \quad S_{p_i}(k) \).

From (3) passing from \( k \) to \( k+1 \), we deduce

\[ A_k(S_{p_i}) \leq i A_k(S_{p_i}) \quad (4) \]

Taking into account the proposition 3. from (4) it results that when we pass from \( k \) to \( k+1 \) we obtain
\[ A_k(S_p) \leq i, p < q \quad \text{and} \quad \sum_{k=1}^{n} A_k(S_p) \leq \sum_{k=1}^{n} A_k(S_q) \quad (5) \]

Because we have

\[ S_p(n) = S_p(1) + \sum_{k=1}^{n} A_k(S_p) \leq S_p(1) + \sum_{k=1}^{n} A_k(S_p) \]

and

\[ S_q(n) = S_q(1) + \sum_{k=1}^{n} A_k(S_q) \]

from (3) and (5) it results

\[ S_p(n) \leq S_q(n) \quad , \quad n \in \mathbb{N}^* \]

8. Proposition. If \( p \) is a prime number then \( S_n < S_p \) for every \( n < p \).

Proof. If \( n \) is a prime number from \( n < p \), using the proposition 3 it results \( S_n(k) < S_p(k) \) for \( k \in \mathbb{N}^* \). If \( n \) is a composed, that is \( n = p_1 \cdot \ldots \cdot p_i \) then \( S_n(k) = \max \{ S_i(k) \} = S_i(k) \).

Because \( n < p \) it results \( p_f < p \) and using the proposition 9 and knowing that \( i p_f \leq p_f < p \) it results that \( S_i(k) \leq S_p(k) \)

therefore

\[ S_n(k) < S_p(k) \quad , \quad k \in \mathbb{N}^* \]

References
