ON SOME RECURRENCE TYPE SMARANDACHE SEQUENCES

A.A.K. MAJUMDAR and H. GUNARTO
Ritsumeikan Asia-Pacific University, 1-1 Jumonjibaru, Beppu-shi 874–8577, Oita-ken, Japan.

ABSTRACT. In this paper, we study some properties of ten recurrence type Smarandache sequences, namely, the Smarandache odd, even, prime product, square product, higher-power product, permutation, consecutive, reverse, symmetric, and pierced chain sequences.

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1. INTRODUCTION

This paper considers the following ten recurrence type Smarandache sequences.

(1) Smarandache Odd Sequence : The Smarandache odd sequence, denoted by $\{OS(n)\}^{\infty}_{n=1}$, is defined by (Ashbacher [1])

$$OS(n)=\overline{135\ \dots (2n-1)},\ n\geq 1. \tag{1.1}$$

A first few terms of the sequence are

1, 13 135, 1357, 13579, 1357911, 135791113, 13579111315, ...

(2) Smarandache Even Sequence : The Smarandache even sequence, denoted by $\{ES(n)\}^{\infty}_{n=1}$, is defined by (Ashbacher [1])

$$ES(n)=\overline{24\dots(2n)},\ n\geq 1. \tag{1.2}$$

A first few terms of the sequence are

2, 24, 246, 2468, 246810, 24681012, 2468101214, ...,

of which only the first is a prime number.

(3) Smarandache Prime Product Sequence : Let $\{p_n\}^{\infty}_{n=1}$ be the (infinite) sequence of primes in their natural order, so that $p_1=2$, $p_2=3$, $p_3=5$, $p_4=7$, $p_5=11$, $p_6=13$, ...

The Smarandache prime product sequence, denoted by $\{PPS(n)\}^{\infty}_{n=1}$, is defined by (Smarandache [2])

$$PPS(n)=p_1p_2\dots p_n+1,\ n\geq 1. \tag{1.3}$$

(4) Smarandache Square Product Sequences : The Smarandache square product sequence of the first kind, denoted by $\{SPS_1(n)\}^{\infty}_{n=1}$, and the Smarandache square product sequence of the second kind, denoted by $\{SPS_2(n)\}^{\infty}_{n=1}$, are defined by (Russo [3])

$$SPS_1(n)=(1^2)(2^2)\dots(n^2)+1=(n!)^2+1,\ n\geq 1, \tag{1.4a}$$

$$SPS_2(n)=(1^2)\ (2^2)\dots(n^2)-1=(n!)^2-1,\ n\geq 1. \tag{1.4b}$$

A first few terms of the sequence $\{SSP_1(n)\}^{\infty}_{n=1}$ are

$SPS_1(1)=2$, $SPS_1(2)=5$, $SPS_1(3)=37$, $SPS_1(4)=577$, $SPS_1(5)=14401$,
$SPS_1(6)=518401=13\times 39877$, $SPS_1(7)=25401601=101\times 251501$,
$SPS_1(8)=1625702401=17\times 95629553$, $SPS_1(9)=131681894401$,

of which the first five terms are each prime.

A first few terms of the sequence $\{SPS2(n)\}^{\infty}_{n=1}$ are

$SPS_2(1)=0$, $SPS_2(2)=3$, $SPS_2(3)=35$, $SPS_2(4)=575$, $SPS_2(5)=14399$,
$SPS_2(6)=518399$, $SPS_2(7)=25401599$, $SPS_2(8)=1625702399$, $SPS_2(9)=131681894399$,

of which, disregarding the first term, the second term is prime, and the remaining terms of the sequence are all composite numbers (see Theorem 6.3).

(5) Smarandache Higher Power Product Sequences : Let m (>3) be a fixed integer. The Smarandache higher power product sequence of the first kind, denoted by, $\{HPPS_1(n)\}^{\infty}_{n=1}$, and the Smarandache higher power product sequence of the second kind, denoted by, $HPPS_2(n)\}^{\infty}_{n=1}$, are defined by

$$HPPS_1(n)=(1^m)(2^m)\ldots(n^m)+1=(n!)^m+1,\ n\geq 1, \qquad (1.5a)$$

$$HPPS_2(n)=(1^m)(2^m)\ldots(n^m)-1=(n!)^m-1,\ n\geq 1. \qquad (1.5b)$$

(6) Smarandache Permutation Sequence : The Smarandache permutation sequence, denoted by $\{PS(n)\}^{\infty}_{n=1}$, is defined by (Dumitrescu and Seleacu [4])

$$PS(n)=\overline{135\ldots(2n-1)(2n)(2n-2)\ldots 42},\ n\geq 1. \qquad (1.6)$$

A first few terms of the sequence are

12, 1342, 135642, 13578642, 13579108642, ...

(7) Smarandache Consecutive Sequence : The Smarandache consecutive sequence, denoted by $\{CS(n)\}^{\infty}_{n=1}$, is defined by (Dumitrescu and Seleacu [4])

$$CS(n)=\overline{123\ldots(n-1)n},\ n\geq 1. \qquad (1.7)$$

A first few terms of the sequence are

1, 12, 123, 1234, 12345, 123456, ...

(8) Smarandache Reverse Sequence : The Smarandache reverse sequence, denoted by, $\{RS(n)\}^{\infty}_{n=1}$, is defined by (Ashbacher [1])

$$RS(n)=\overline{n(n-1)\ldots 21},\ n\geq 1. \qquad (1.8)$$

A first few terms of the sequence are

1, 21, 321, 4321, 54321, 654321, ...

(9) Smarandache Symmetric Sequence: The Smarandache symmetric sequence, denoted by $\{SS(n)\}^{\infty}_{n=1}$, is defined by (Ashbacher [1])

1,11,121, 12321, 1234321, 123454321, 12345654321, ...

Thus,

$$SS(n)=\overline{12\ldots(n-2)(n-1)(n-2)\ldots 21},\ n\geq 3; \quad SS(1)=1,\ SS(2)=11. \qquad (1.9)$$

(10) Smarandache Pierced Chain Sequence : The Smarandache pierced chain sequence, denoted by $\{PCS(n)\}_{n=1}^{\infty}$, is defined by (Ashbacher [1])

101, 1010101, 10101010101, 101010101010101, ..., (1.10)

which is obtained by successively concatenating the string 0101 to the right of the preceding terms of the sequence, starting with PCS(1)=101.

As has been pointed out by Ashbacher, all the terms of the sequence $\{PCS(n)\}_{n=1}^{\infty}$ is divisible by 101. We thus get from the sequence $\{PCS(n)\}_{n=1}^{\infty}$, on dividing by 101, the sequence $\{PCS(n)/101\}_{n=1}^{\infty}$. The elements of the sequence $\{PCS(n)/101\}_{n=1}^{\infty}$ are

1, 10001, 100010001, 1000100010001, ... (1.11)

Smarandache [5] raised the question : How many terms of the sequence $\{PCS(n)/101\}_{n=1}^{\infty}$ are prime?

In this paper, we consider some of the properties satisfied by these ten Smarandache sequences in the next ten sections where we derive the recurrence relations as well.

For the Smarandache odd, even, consecutive and symmetric sequences, Ashbacher [1] raised the question : Are there any Fibonacci or Lucas numbers in these sequences?

We recall that the sequence of Fibonacci numbers, $\{F(n)\}_{n=1}^{\infty}$, and the sequence of Lucas numbers $\{L(n)\}_{n=1}^{\infty}$, are defined by (Ashbacher [1])

$$F(0)=0,\ F(1)=1;\ F(n+2)=F(n+1)+F(n),\ n\geq 0, \qquad (1.12)$$

$$L(0)=2,\ L(1)=1;\ L(n+2)=L(n+1)+L(n),\ n\geq 0, \qquad (1.13)$$

Based on computer search for Fibonacci and Lucas numbers, Ashbacher conjectures that there are no Fibonacci or Lucas numbers in any of the Smarandache odd, even, consecutive and symmetric sequences (except for the trivial cases). This paper confirms the conjectures of Ashbacher. We prove further that none of the Smarandache prime product and reverse sequences contain Fibonacci or Lucas numbers (except for the trivial cases).

For the Smarandache even, prime product, permutation and square product sequences, the question is : Are there any perfect powers in each of these sequences? We have a partial answer for the first of these sequences, while for each of the remaining sequences, we prove that no number can be expressed as a perfect power. We also prove that no number of the Smarandache higher power product sequences is square of a natural number.

For the Smarandache odd, prime product, consecutive, reverse and symmetric sequences, the question is : How many primes are there in each of these sequences? For the Smarandache even sequence, the question is : How many elements of the sequence are twice a prime? These questions still remain open.

In the subsequent analysis, we would need the following result.

Lemma 1.1 : $3|(10^m+10^n+1)$ for all integers $m,n\geq 0$.

Proof : We consider the following three possible cases separately :

(1) $m=n=0$. In this case, the result is clearly true.

(2) $m=0$, $n\geq 1$. Here,

$$10^m+10^n+1=10^n+2=(10^n-1)+3,$$

and so the result is true, since $3|10^n-1=9(1+10+10^2+\ldots+10^{n-1})$.

(3) $m\geq 1$, $n\geq 1$. In this case, writing

$$10^m+10^n+1=(10^m-1)+(10^n-1)+3,$$

we see the validity of the result. □

2. SMARANDACHE ODD SEQUENCE $\{OS(n)\}_{n=1}^{\infty}$

The Smarandache odd sequence is the sequence of numbers formed by repeatedly concatenating the odd positive integers, and the n-th term of the sequence is given by (1.1).

For any $n\geq 1$, $OS(n+1)$ can be expressed in terms of $OS(n)$ as follows : For $n\geq 1$,

$$\begin{aligned} OS(n+1) &= \overline{135\ldots(2n-1)(2n+1)} \\ &= 10^s OS(n)+(2n+1) \quad \text{for some integer } s\geq 1. \end{aligned} \tag{2.1}$$

More precisely,

s=number of digits in $(2n+1)$.

Thus, for example, $OS(5)=(10)OS(4)+7$, while, $OS(6)=(10^2)OS(5)+11$.

By repeated application of (2.1), we get

$$OS(n+3)=10^s\, OS(n+2)+(2n+5) \text{ for some integer } s\geq 1$$

$$=10^s[10^t\, OS(n+1)+(2n+3)]+(2n+5) \text{ for some integer } t\geq 1 \tag{2.2a}$$

$$=10^{s+t}[10^u\, OS(n)+(2n+1)]+(2n+3)10^s+(2n+5) \text{ for some integer } u\geq 1, \tag{2.2b}$$

so that

$$OS(n+3)=10^{s+t+u}OS(n)+(2n+1)10^{s+t}+(2n+3)10^s+(2n+5), \tag{2.3}$$

where $s\geq t\geq u\geq 1$.

Lemma 2.1 : $3 \mid OS(n)$ if and only if $3 \mid OS(n+3)$.

Proof : For any s, t with $s\geq t\geq 1$, by Lemma 1.1,

$$3 \mid [(2n+1)10^{s+t}+(2n+3)10^s+(2n+5)]=(2n+1)(10^{s+t}+10^s+1)+(10^s+2).$$

The result is now evident from (2.3). □

From the expression of OS(n+3) given in (2.2), we see that, for all $n\geq1$,

$$OS(n+3)=10^{s+t}\,OS(n+1)+\overline{(2n+3)(2n+5)}$$

$$=10^{s+t+u}\,OS(n)+\overline{(2n+1)(2n+3)(2n+5)}.$$

The following result has been proved by Ashbacher [1].

Lemma 2.2 : $3\mid OS(n)$ if and only if $3\mid n$. In particular, $3\mid OS(3n)$ for all $n\geq1$.

In fact, it can be proved that $9|OS(3n)$ for all $n\geq1$.

We now prove the following result.

Lemma 2.3 : $5\mid OS(5n+3)$ for all $n\geq0$.

Proof : From (2.1), for any arbitrary but fixed $n\geq0$,

$$OS(5n+3)=10^{s}\,OS(5n+2)+(10n+5) \text{ for some integer } s\geq1.$$

The r.h.s. is clearly divisible by 5, and hence $5\mid OS(5n+3)$.

Since n is arbitrary, the lemma is established. □

Ashbacher [1] devised a computer program which was then run for all numbers from 135 up through OS(2999)=135...29972999, and based on the findings, he conjectures that (except for the trivial case of n=1, for which OS(1)=1=F(1)=L(1)) there are no numbers in the Smarandache odd sequence that are also Fibonacci (or, Lucas) numbers. In Theorem 2.1 and Theorem 2.2, we prove the conjectures of Ashbacher in the affirmative. The proof of the theorems relies on the following results.

Lemma 2.4 : For any $n\geq1$, $OS(n+1)>10\,OS(n)$.

Proof : From (2.1), for any $n\geq1$,

$$OS(n+1)=10^{s}\,OS(n)+(2n+1)>10^{s}\,OS(n)>10\,OS(n),$$

where $s\geq1$ is an integer. We thus get the desired inequality. □

Corollary 2.1 : For any $n\geq1$, $OS(n+2)-OS(n)>9[OS(n+1)+OS(n)]$.

Proof : From Lemma 2.4,

$$OS(n+1)-OS(n)>9\,OS(n) \text{ for all } n\geq1. \tag{2.4}$$

Now, using the inequality (2.4), we get

$$OS(n+2)-OS(n)=[OS(n+2)-OS(n+1)]+[OS(n+1)-OS(n)]>9[OS(n+1)+OS(n)],$$

which establishes the lemma. □

Theorem 2.1 : (Except for n=1,2 for which OS(1)=1=F(1)=F(2), OS(2)=13=F(7)) there are no numbers in the Smarandache odd sequence that are also Fibonacci numbers.

Proof : Using Corollary 2.1, we see that, for all $n\geq1$,

$$OS(n+2)-OS(n)>9[OS(n+1)+OS(n)]>OS(n+1). \tag{2.5}$$

Thus, no numbers of the Smarandache odd sequence satisfy the recurrence relation (2.10) satisfied by the Fibonacci numbers. □

By similar reasoning, we have the following result.

Theorem 2.2 : (Except for n=1 for which OS(1)=1=L(2)) there are no numbers in the Smarandache odd sequence that are Lucas numbers.

Searching for primes in the Smarandache odd sequence (using UBASIC program), Ashbacher [1] found that among the first 21 elements of the sequence, only OS(2), OS(10) and OS(16) are primes. Marimutha [6] conjectures that there are infinitely many primes in the Smarandache odd sequence, but the conjecture still remains to be resolved.

In order to search for primes in the Smarandache odd sequence, by virtue of Lemma 2.2 and Lemma 2.3, it is sufficient to check the terms of the forms $OS(3n\pm1)$, $n\geq1$, where neither $3n+1$ nor $3n-1$ is of the form $5k+3$ for some integer $k\geq1$.

3. SMARANDACHE EVEN SEQUENCE $\{ES(n)\}_{n=1}^{\infty}$

The Smarandache even sequence, whose n-th term is given by (1.2), is the sequence of numbers formed by repeatedly concatenating the even positive integers.

We note that, for any $n\geq 1$,

$$ES(n+1)=\overline{24\ldots(2n)(2n+2)}$$
$$=10^{s}\,ES(n)+(2n+2) \text{ for some integer } s\geq 1. \qquad (3.1)$$

More precisely,

$$s = \text{number of digits in } (2n+2).$$

Thus, for example, $ES(4)=2468=10\,ES(3)+8$, while, $ES(5)=246810=10^2\,ES(4)+10$.

From (3.1), the following result follows readily.

Lemma 3.1: For any $n\geq 1$, $ES(n+1)>10\,ES(n)$.

Using Lemma 3.1, we can prove that

$$ES(n+2)-ES(n)>9[ES(n+1)+ES(n)] \text{ for all } n\geq 1. \qquad (3.2)$$

The poof is similar to that given in establishing the inequality (2.1) and is omitted here.

By repeated application of (3.1), we see that, for any $n\geq 1$,

$$ES(n+2)=10^{t}\,ES(n+1)+(2n+4) \text{ for some integer } t\geq 1$$
$$=10^{t}[10^{u}\,ES(n)+(2n+2)]+(2n+4) \text{ for some integer } u\geq 1$$
$$=10^{u+t}\,ES(n)+(2n+2)10^{t}+(2n+4),$$

so that

$$ES(n+3)=10^{s}\,ES(n+2)+(2n+6) \text{ for some integer } s\geq 1$$
$$=10^{s}[10^{t}\,ES(n+1)+(2n+4)]+(2n+6)$$
$$=10^{s+t+u}ES(n)+(2n+2)10^{s+t}+(2n+2)10^{s}+(2n+6), \qquad (3.3)$$

for some integers s, t and u with $s\geq t\geq u\geq 1$.

From (3.3), we see that

$$ES(n+3)=10^{s+t}\,ES(n+1)+\overline{(2n+4)(2n+6)}$$

$$=10^{s+t+u}\,ES(n)+\overline{(2n+2)(2n+4)(2n+6)}.$$

Using (3.3), we can prove the following result.

Lemma 3.2 : If $3\mid ES(n)$ for some $n\geq 1$, then $3\mid ES(n+3)$, and conversely.

Lemma 3.3 : For all $n\geq 1$, $3\mid ES(3n)$.

Proof : The proof is by induction on n. Since $ES(3)=246$ is divisible by 3, the lemma is true for n=1. We now assume that the result is true for some n, that is, $3\mid ES(3n)$ for some n.

Now, by Lemma 3.2, together with the induction hypothesis, we see that $ES(3n+3)=ES(3(n+1))$ is divisible by 3. Thus the result is true for n+1. □

Corollary 3.1 : For all $n\geq 1$, $3\mid ES(3n-1)$.

Proof : Let n (≥ 1) be any arbitrary but fixed integer. From (3.1),

$$ES(3n)=10^{s}\,ES(3n-1)+(6n) \text{ for some integer } s\geq 1.$$

Now, by Lemma 3.2, $3\mid ES(3n)$. Therefore, 3 must also divide $ES(3n-1)$.

Since n is arbitrary, the lemma is proved. □

Corollary 3.2 : For any $n\geq 1$, $3\nmid ES(3n+1)$.

Proof : Let n (≥ 1) be any arbitrary but fixed integer. From (3.1),

$$ES(3n+1)=10^{s}ES(3n)+(6n+2) \text{ for some integer } s\geq 1.$$

Since $3\mid ES(3n)$, but 3 does not divide $(6n+2)$, the result follows. □

Lemma 3.4 : $4 \mid ES(2n)$ for all $n \geq 1$.

Proof : Since $4 \mid ES(2)=24$ and $4 \mid ES(4)=2468$, we see that the result is true for n=1,2. Now, from (3.1), for $n \geq 1$,

$$ES(2n)=10^s\, ES(2n-1)+(4n),$$

where s is the number of digits in (4n). Clearly, $s \geq 2$ for all $n \geq 3$. Thus, $4 \mid 10^s$ if $n \geq 3$, and we get the desired result. □

Corollary 3.3 : For any $n \geq 0$, $4 \nmid ES(2n+1)$.

Proof : Clearly the result is true for n=0, since ES(1)=2 is not divisible by 4. For $n \geq 1$, from (3.1),

$$ES(2n+1)=10^s\, ES(2n)+(4n+2) \text{ for some integer } s \geq 1.$$

By Lemma 3.4, $4 \mid ES(2n)$. Since $4 \nmid (4n+2)$, the result follows. □

Lemma 3.5 : For all $n \geq 1$, $10 \mid ES(5n)$.

Proof : For any arbitrary but fixed $n \geq 1$, from (3.1),

$$ES(5n)=10^s\, ES(5n-1)+(10n) \text{ for some integer } s \geq 1.$$

The result is now evident from the above expression of ES(5n). □

Corollary 3.4 : $20 \mid ES(10n)$ for all $n \geq 1$.

Proof : follows by virtue of Lemma 3.4 and Lemma 3.5. □

Based on the computer findings with numbers up through ES(1499)=2468...29962998, Ashbacher [1] conjectures that (except for the case of ES(1)=2=F(3)=L(0)) there are no numbers in the Smarandache even sequence that are also Fibonacci (or, Lucas) numbers. The following two theorems establish the validity of Ashbacher's conjectures. The proofs of the theorems make use of the inequality (3.2) and are similar to those used in proving Theorem 2.1. We thus omit the proof here.

Theorem 3.1 : (Except for ES(1)=2=F(3)) there are no numbers in the Smarandache even sequence that are Fibonacci numbers.

Theorem 3.2 : (Except for ES(1)=2=L(0)) there are no numbers in the Smarandache even sequence that are Lucas numbers.

Ashbacher [1] raised the question: Are there any perfect powers in ES(n)? The following theorem gives a partial answer to the question.

Theorem 3.3 : None of the terms of the subsequence $\{ES(2n-1)\}_{n=1}^{\infty}$ is a perfect square or higher power of an integer (>1).

Proof : Let, for some $n \geq 1$,

$$ES(n)=\overline{24\ldots(2n)}=x^2 \text{ for some integer } x>1.$$

Now, since ES(n) is even for all $n \geq 1$, x must be even. Let x=2y for some integer $y \geq 1$. Then,

$$ES(n)=(2y)^2=4y^2,$$

which shows that $4 \mid ES(n)$.

Now, if n is odd of the form 2k−1, $k \geq 1$, by Corollary 3.3, ES(2k−1) is not divisible by 4, and hence numbers of the form ES(2k−1), $k \geq 1$, can not be perfect squares. By same reasoning, none of the terms ES(2n−1), $n \geq 1$, can be expressed as a cube or higher powers of an integer. □

Remark 3.1 : It can be seen that, if n is of the form $k \times 10^s+4$ or $k \times 10^s+6$, where k ($1 \leq k \leq 9$) and s (≥ 1) are integers, then ES(n) cannot be a perfect square (and hence, cannot be any even power of a natural number). The proof is as follows : If

$$ES(n)=x^2 \text{ for some integer } x>1, \qquad (*)$$

then x must be an even integer. The following table gives the possible trailing digits of x and the corresponding trailing digits of x^2 :

Trailing digit of x	Trailing digit of x^2
2	4
4	6
6	6
8	4

Since the trailing digit of $ES(k\times10^s+4)$ is 8 for all admissible values of k and s, it follows that the representation of $ES(k\times10^s+4)$ in the form (*) is not possible. By similar reasoning, if n is of the form $n=k\times10^s+6$, then $ES(n)=ES(k\times10^s+6)$ with the trailing digit of 2, cannot be expressed as a perfect square (and hence, any even power of a natural number). Thus, it remains to consider the cases when n is one of the forms (1) $n=k\times10^s$, (2) $n=k\times10^s+2$, (3) $n=k\times10^s+8$ (where, in all the three cases, k ($1\le k\le9$) and s (≥1) are integers). Smith [7] conjectures that none of the terms of the sequence $\{ES(n)\}^{\infty}_{n=1}$ is a perfect power.

4. SMARANDACHE PRIME PRODUCT SEQUENCE $\{PPS(n)\}^{\infty}_{n=1}$

The n-th term, PPS(n), of the Smarandache prime product sequence is given by (1.3). The following lemma gives a recurrence relation in connection with the sequence.

Lemma 4.1: $PPS(n+1)=p_{n+1}\,PPS(n)-(p_{n+1}-1)$ for all $n\ge1$.

Proof : By definition,

$$PPS(n+1)=p_1p_2\ldots p_np_{n+1}+1=(p_1p_2\ldots p_n+1)p_{n+1}-p_{n+1}+1,$$

which now gives the desired relationship. □

From Lemma 4.1, we get

Corollary 4.1: $PPS(n+1)-PPS(n)=[PPS(n)-1](p_{n+1}-1)$ for all $n\ge1$.

Lemma 4.2 : (1) $PPS(n)<(p_n)^{n-1}$ for all $n\ge4$, (2) $PPS(n)<(p_n)^{n-2}$ for all $n\ge7$, (3) $PPS(n)<(p_n)^{n-3}$ for all $n\ge10$, (4) $PPS(n)<(p_{n+1})^{n-1}$ for all $n\ge3$, (5) $PPS(n)<(p_{n+1})^{n-2}$ for all $n\ge6$, (6) $PPS(n)<(p_{n+1})^{n-3}$ for all $n\ge9$.

Proof : We prove parts (3) and (6) only, the proof of the other parts is similar.

To prove part (3) of the lemma, we note that the result is true for n=10, since

$$PPS(10)=6469693231<(p_{10})^7=29^7=17249876309.$$

Now, assuming the validity of the result for some integer k (≥10), and using Lemma 4.1, we see that,

$$\begin{aligned}PPS(k+1)=p_{k+1}\,PPS(k)-(p_{k+1}-1) &<p_{k+1}\,PPS(k)\\ &<p_{k+1}(p_k)^{n-3}\text{ (by the induction hypothesis)}\\ &<(p_{k+1})(p_{k+1})^{n-3}=(p_{k+1})^{n-2},\end{aligned}$$

where the last inequality follows from the fact that the sequence of primes, $\{p_n\}^{\infty}_{n=1}$, is strictly increasing in n (≥1). Thus, the result is true for k+1 as well.

To prove part (6) of the lemma, we note that the result is true for n=9, since

$$PPS(9)=223092871<(p_{10})^6=29^6=594823321.$$

Now to appeal to the principle of induction, we assume that the result is true for some integer k (≥9). Then using Lemma 4.1, together with the induction hypothesis, we get

$$PPS(k+1)=p_{k+1}\,PPS(k)-(p_{k+1}-1)<p_{k+1}\,PPS(k)<p_{k+1}(p_{k+1})^{k-3}=(p_{k+1})^{k-2}.$$

Thus the result is true for k+1.

All these complete the proof by induction. □

Lemma 4.3 : Each of PPS(1), PPS(2), PPS(3), PPS(4) and PPS(5) is prime, and for $n\ge6$, PPS(n) has at most n–4 prime factors, counting multiplicities.

Proof : Clearly PPS(1)=3, PPS(2)=7, PPS(3)=31, PPS(4)=211, PPS(5)=2311 are all primes. Also, since

PPS(6)=30031=59×509, PPS(7)=510511=19×97×277, PPS(8)=9699691=347×27953, we see that the lemma is true for $6 \leq n \leq 8$.

Now, if p is a prime factor of PPS(n), then $p \geq p_{n+1}$. Therefore, if for some $n \geq 9$, PPS(n) has n–3 (or more) prime factors (counted with multiplicity), then $PPS(n) \geq (p_{n+1})^{n-3}$, contradicting part (6) of Lemma 4.2.

Hence the lemma is established. □

Lemma 4.3 above improves the earlier results (Prakash [8], and Majumdar [9]).

The following lemma improves a previous result (Majumdar [10]).

Lemma 4.4 : For any $n \geq 1$ and $k \geq 1$, PPS(n) and PPS(n+k) can have at most k–1 number of prime factors (counting multiplicities) in common.

Proof : For any $n \geq 1$ and $k \geq 1$,

$$PPS(n+k) - PPS(n) = p_1 p_2 \ldots p_n (p_{n+1} p_{n+2} \ldots p_{n+k} - 1). \tag{4.1}$$

If p is a common prime factor of PPS(n) and PPS(n+k), since $p \geq p_{n+k}$, it follows from (4.1) that $p \mid (p_{n+1} p_{n+2} \ldots p_{n+k} - 1)$. Now if PPS(n) and PPS(n+k) have k (or more) prime factors in common, then the product of these common prime factors is greater than $(p_{n+k})^k$, which can not divide $p_{n+1} p_{n+2} \ldots p_{n+k} - 1 < (p_{n+k})^k$.

This contradiction proves the lemma. □

Corollary 4.2 : For any integers n (≥ 1) and k (≥ 1), if all the prime factors of $p_{n+1} p_{n+2} \ldots p_{n+k} - 1$ are less than p_{n+k}, then PPS(n) and PPS(n+k) are relatively prime.

Proof : If p is any common prime factor of PPS(n) and PPS(n+k), then $p \mid (p_{n+1} p_{n+2} \ldots p_{n+k} - 1)$. Also, such $p > p_{n+k}$, contradicting the hypothesis of the corollary. Thus, if all the common prime factors of PPS(n) and PPS(n+k) are less than p_{n+k}, then (PPS(n),PPS(n+k)=1. □

The following result has been proved by others (Prokash [8] and Majumdar [10]). Here we give a simpler proof.

Theorem 4.1: For any $n \geq 1$, PPS(n) is never a square or higher power of an integer (>1).

Proof : Clearly, none of PPS(1), PPS(2), PPS(3), PPS(4) and PPS(5) can be expressed as powers of integers (by Lemma 4.3).

Now, if possible, let for some $n \geq 6$,

$$PPS(n) = x^{\ell} \text{ for some integers } x\ (>3),\ \ell\ (\geq 2). \tag{*}$$

Without loss of generality, we may assume that ℓ is a prime (if ℓ is a composite number, letting $\ell = pr$ where p is prime, we have $PPS(n) = (x^r)^p = N^p$, where $N = x^r$). By Lemma 4.3, $\ell \leq n-4$ and so ℓ cannot be greater than p_{n-5} ($\ell \geq p_{n-4} \Rightarrow \ell > n-4$, since $p_n > n$ for all $n \geq 1$). Hence, ℓ must be one of the primes $p_1, p_2, \ldots, p_{n-5}$. Also, since PPS(n) is odd, x must be odd. Let $x = 2y+1$ for some integer $y > 0$. Then, from (*),

$$p_1 p_2 \ldots p_n = (2y+1)^{\ell} - 1$$

$$= (2y)^{\ell} + \binom{\ell}{1}(2y)^{\ell-1} + \ldots + \binom{\ell}{\ell-1}(2y). \tag{**}$$

If $\ell = 2$, we see from (**), $4 \mid p_1 p_2 \ldots p_n$, which is absurd. On the other hand, for $\ell \geq 3$, since $\ell \mid p_1 p_2 \ldots p_n$, it follows from (**) that $\ell \mid y$, and consequently, $\ell^2 \mid p_1 p_2 \ldots p_n$, which is impossible.

Hence, the representation of PPS(n) in the form (*) is not possible. □

Using Corollary 4.1 and the fact that PPS(n+1)–PPS(n)>0, we get

$$\begin{aligned} PPS(n+2) - PPS(n) &= [PPS(n+2) - PPS(n+1)] + [PPS(n+1) - PPS(n)] \\ &> [PPS(n+1) - 1](p_{n+2} - 1) \\ &> 2[PPS(n+1) - 1] \text{ for all } n \geq 1. \end{aligned}$$

Hence,

$$PPS(n+2)-PPS(n)>PPS(n+1) \text{ for all } n\geq 1. \qquad (4.2)$$

The inequality (4.2) shows that no elements of the Smarandache prime product sequence satisfy the recurrence relation for Fibonacci (or, Lucas) numbers. This leads to the following theorem.

Theorem 4.2 : There are no numbers in the Smarandache prime product sequence that are Fibonacci (or Lucas) numbers (except for the trivial cases of $PPS(1)=3=F(4)=L(2)$, $PPS(2)=7=L(4)$).

5. SMARANDACHE SQUARE PRODUCT SEQUENCES $\{SPS_1(n)\}_{n=1}^{\infty}$, $\{SPS_2(n)\}_{n=1}^{\infty}$

The n-th terms, $SPS_1(n)$ and $SPS_2(n)$, are given in (1.4a) and (1.4b) respectively.

In Theorem 5.1, we prove that, for any $n\geq 1$, neither of $SPS_1(n)$ and $SPS_2(n)$ is a square of an integer (>1). To prove the theorem, we need the following results.

Lemma 5.1: The only non–negative integer solution of the Diophantine equation $x^2-y^2=1$ is $x=1, y=0$.

Proof : The given Diophantine equation is equivalent to $(x-y)(x+y)=1$, where both $x-y$ and $x+y$ are integers. Therefore, the only two possibilities are

(1) $x-y=1=x+y$, (2) $x-y=-1=x+y$,

the first of which gives the desired non-negative solution. □

Corollary 5.1: Let N (>1) be a fixed number. Then,

(1) The Diphantine equation $x^2-N=1$ has no (positive) integer solution x,

(2) The Diophantine equation $N-y^2=1$ has no (positive) integer solution y.

Theorem 5.1 : For any $n\geq 1$, none of $SPS_1(n)$ and $SPS_2(n)$ is a square of an integer (>1).

Proof : If possible, let

$$SPS_1(n)\equiv(n\,!)^2+1=x^2 \text{ for some integers } n\geq 1,\ x>1.$$

But, by Corollary 5.1(1), this Diophantine equation has no integer solution x.

Again, if

$$SPS_2(n)\equiv(n\,!)^2-1=y^2 \text{ for some integers } n\geq 1,\ y>1,$$

then, by Corollary 5.1(2), this Diophantine equation has no integer solution y.

All these complete the proof of the theorem. □

In Theorem 5.2, we prove a stronger result, for which we need the results below.

Lemma 5.2 : Let m (≥2) be a fixed integer. Then, the only non-negative integer solution of the Diophantine equation $x^2+1=y^m$ is $x=0, y=1$.

Proof : For m=2, the result follows from Lemma 5.1. So, it is sufficient to consider the case when m>2. However, we note that it is sufficient to consider the case when m is odd; if m is even, say, m=2q for some integer q>1, then rewriting the given Diophantine equation as $(y^q)^2-x^2=1$, we see that, by Lemma 5.1, the only non-negative integer solution is $y^q=1$, $x=0$, that is $x=0$, $y=1$, as required.

So, let m be odd, say, m=2q+1 for some integer q≥1. Then, the given Diophantine equation can be written as

$$x^2=y^{2q+1}-1=(y-1)(y^{2q}+y^{2q-1}+...+1). \qquad (***)$$

From (***), we see that x=0 if and only if y=1, since $y^{2q}+y^{2q-1}+...+1>0$.

Now, if $x\neq 0$, from (***), the only two possibilities are

(1) $y-1=x$, $y^{2q}+y^{2q-1}+...+1=x$.

But then $y=x+1$, and we are led to the equation $(x+1)^{2q}+(x+1)^{2q-1}+...+(x+1)^2+2=0$, which is impossible.

(2) $y-1=1$, $y^{2q}+y^{2q-1}+...+1=x^2$.

Then, $y=2$ together with the equation

$$x^2=2^{2q+1}-1. \qquad (5.1)$$

But the equation (5.1) has no integer solution x (>1). To prove this, we first note that any integer x satisfying (5.1) must be odd. Now rewriting (5.1) in the following equivalent form

$$(x-1)(x+1)=2(2^q-1)(2^q+1),$$

we see that the l.h.s. is divisible by 4, while the r.h.s. is not divisible by 4 since both 2^q-1 and 2^q+1 are odd.

Thus, if $x\neq 0$, then we reach to a contradiction in either of the above cases. This contradiction establishes the lemma. □

Corollary 5.2 : Let m (≥ 2) and N (>0) be two fixed integers. Then, the Diophantine equation $N^2+1=y^m$ has no integer solution y.

Corollary 5.3 : Let m (≥ 2) and N (>1) be two fixed integers. Then, the Diophantine equation $x^2+1=N^m$ has no (positive) integer solution x.

Lemma 5.3 : Let m (≥ 2) be a fixed integer. Then, the only non-negative integer solutions of the Diophantine equation $x^2-y^m=1$ are (1) $x=1$, $y=0$; (2) $x=3$, $y=2$, $m=3$.

Proof : For m=2, the lemma reduces to Lemma 5.1. So we consider the case when $m\geq 3$.

From the given Diophantine equation, we see that, $y=0$ if and only if $x=\pm 1$, giving the only non-negative integer solution $x=1$, $y=0$. To see if the given Diophantine equation has any non-zero integer solution, we assume that $x\neq 1$.

If m is even, say, $m=2q$ for some integer $q\geq 1$, then $x^2-y^m\equiv x^2-(y^q)^2=1$, which has no integer solution y for any $x>1$ (by Corollary 5.1(2)).

Next, let m be odd, say, $m=2q+1$ for some integer $q\geq 1$. Then, $x^2-y^{2q+1}=1$, that is,

$$(x-1)(x+1)=y^{2q+1}.$$

We now consider the following cases that may arise :

(1) $x-1=1$, $x+1=y^{2q+1}$.

Here, $x=2$ together with the equation $y^{2q+1}=3$, which has no integer solution y.

(2) $x-1=y$, $x+1=y^{2q}$.

Rewriting the second equation in the equivalent form $(y^q-1)(y^q+1)=x$, we see that $(y^q+1)\mid x$. But this contradicts the first equation $x=y+1$ if $q>1$, since for $q>1$, $y^q+1>y+1=x$.

If $q=1$, then

$$(y-1)(y+1)=x \Rightarrow y-1=1,\ y+1=x,$$

so that $y=2$, $x=3$, $m=3$, which is a solution of the given Diophantine equation.

(3) $x-1=y^t$ for some integer t with $2\leq t\leq q$, $q\geq 2$ (so that $x+1=y^{2q-t+1}$).

In this case, we have

$$2x=y^t[1+y^{2(q-t)+1}].$$

Since x does not divide y, it follows that

$$1+y^{2(q-t)+1}=Cx \text{ for some integer } C\geq 1.$$

Thus,

$$2x=y^t(Cx) \Rightarrow Cy^t=2.$$

If $C=2$, then $y=1$, and the resulting equation $x^2=2$ has no integer solution. On the other hand, if $C\neq 2$, the equation $Cy^t=2$ has no integer solution. Thus, case (3) cannot occur.

All these complete the proof of the lemma. □

Corollary 5.4 : The only non-negative integer solution of the Diophantine equation $x^2-y^3=1$ is $x=3$, $y=2$.

Corollary 5.5 : Let m (>3) be a fixed integer. Then, the Diophantine equation $x^2-y^m=1$ has $x=1$, $y=0$ as its only non-negative integer solution.

Corollary 5.6 : Let m (>3) and N (>0) be two fixed integers. Then, the Diophantine equation $x^2-N^m=1$ has no integer solution x.

Corollary 5.7 : Let m (≥3) and N (>1) be two fixed integers with N≠3. Then, the Diophantine equation $N^2-y^m=1$ has no integer solution.

We are now in a position to prove the following theorem.

Theorem 5.2 : For any n≥1, none of the $SPS_1(n)$ and $SPS_2(n)$ is a cube or higher power of an integer (>1).

Proof : is by contradiction. Let, for some integer n≥1,

$$SPS_1(n)\equiv(n!)^2+1=y^m \text{ for some integers } y>1,\ m\geq 3.$$

By Corollary 5.2, the above equation has no integer solution y.

Again, if for some integer n≥1,

$$SPS_2(n)\equiv(n!)^2-1=z^s \text{ for some integer } z\geq 1,\ s\geq 3,$$

we have contradiction to Corollary 5.7. □

The following result gives the recurrence relations satisfied by $SPS_1(n)$ and $SPS_2(n)$.

Lemma 5.4 : For all n≥1,

(1) $SPS_1(n+1)=(n+1)^2SPS_1(n)-n(n+2)$,

(2) $SPS_2(n+1)=(n+1)^2SPS_2(n)+n(n+2)$.

Proof : The proof is for part (1) only. Since

$$SPS_1(n+1)=[(n+1)!]^2+1=(n+1)^2[(n!)^2+1]-(n+1)^2+1,$$

the result follows. □

Lemma 5.5 : For all n≥1,

(1) $SPS_1(n+2)-SPS_1(n)>SPS_1(n+1)$,

(2) $SPS_2(n+2)-SPS_2(n)>SPS_2(n+1)$.

Proof : Using Lemma 5.4, it is straightforward to prove that

$$SPS_1(n+2)-SPS_1(n)=SPS_2(n+2)-SPS_2(n)=(n!)^2\ [(n+1)^2(n+2)^2-1].$$

Some algebraic manipulations give the desired inequalities. □

Lemma 5.5 can be used to prove the following results.

Theorem 5.3 : (Except for the trivial cases, $SPS_1(1)=2=F(3)=L(0)$, $SPS_1(2)=5=F(5)$) there are no numbers of the Smarandache square product sequence of the first kind that are Fibonacci (or Lucas) numbers.

Theorem 5.4 : (Except for the trivial cases, $SPS_2(1)=0=F(0)$, $SPS_2(2)=3=F(4)=L(2)$) there are no numbers of the Smarandache square product sequence of the second kind that are Fibonacci (or Lucas) numbers.

The question raised by Iacobescu [11] is : How many terms of the sequence $\{SPS_1(n)\}^{\infty}_{n=1}$ are prime?

The following theorem, due to Le [12], gives a partial answer to the above question.

Theorem 5.5 : If n (>2) is an even integer such that 2n+1 is prime, then $SPS_1(n)$ is not a prime.

Russo [3] gives tables of values of $SPS_1(n)$ and $SPS_2(n)$ for 1≤n≤20. Based on computer results, Russo [3] conjectures that each of the sequences $\{SPS_1(n)\}^{\infty}_{n=1}$ and $\{SPS_2(n)\}^{\infty}_{n=1}$ contains only a finite number of primes.

6. SMARANDACHE HIGHER POWER PRODUCT SEQUENCES $\{HPPS_1(n)\}^{\infty}_{n=1}$, $\{HPPS_2(n)\}^{\infty}_{n=1}$

The n-th terms of the Smarandache higher power product sequences are given in (1.5). The following lemma gives the recurrence relation satisfied by $HPPS_1(n)$ and $HPPS_2(n)$.

Lemma 6.1 : For all $n \geq 1$,
(1) $HPPS_1(n+1)=(n+1)^m HPPS_1(n)-[(n+1)^m+1]$,
(2) $HPPS_2(n+1)=(n+1)^m HPPS_2(n)+[(n+1)^m+1]$.
Theorem 6.1: For any integer $n \geq 1$, none of $HPPPS_1(n)$ and $HPPS_2(n)$ is a square of an integer (>1).
Proof : If possible, let

$$HPPS_1(n)\equiv(n!)^m+1=x^2 \text{ for some integer } x>1.$$

This leads to the Diophantine equation $x^2-(n!)^m=1$, which has no integer solution x, by virtue of Corollary 5.6 (for m>3). Thus, if m>3, $HPPS_1(n)$ cannot be a square of a natural number (>1) for any $n \geq 1$.

Next, let, for some integer $n \geq 2$ (HPPS2(1)=0)

$$HPPS_2(n)\equiv(n!)^m-1=y^2 \text{ for some integer } y \geq 1.$$

Then, we have the Diophantine equation $y^2+1=(n!)^m$, and by Corollary 5.3, it has no integer solution y. Thus, $HPPS_2(n)$ cannot be a square of an integer (>1) for any $n \geq 1$. □
The following two theorems are due to Le [13,14].
Theorem 6.2: If m is not a number of the form 2^ℓ for some $\ell \geq 1$, then the sequence $\{HPPS_1(n)\}^{\infty}_{n=1}$ contains only one prime, namely, $HPPPS_1(1)=2$.
Theorem 6.3: If both m and 2^m-1 are primes, then the sequence $\{HPPS_2(n)\}^{\infty}_{n=1}$ contains only one prime, $HPPS_2(2)= 2^m-1$; otherwise, the sequence does not contain any prime.
Remark 6.1 : We have defined the Smarandache higher power product sequences under the restriction that m>3, and under such restriction, as has been proved in Theorem 6.1, none of $HPPS_1(n)$ and $HPPS_2(n)$ is a square of an integer (>1) for any $n \geq 1$. However, if m=3, the situation is a little bit different : For any $n \geq 1$, $HPPS_2(n)=(n!)^3-1$ still cannot be a perfect square of an integer (>1), by virtue of Corollary 5.3, but since $HPPS_1(n)=(n!)^3+1$, we see that $HPPS_1(2)=(2!)^3+1=3^2$, that is, $HPPS_1(2)$ is a perfect square. However, this is the only term of the sequence $\{SPPS_1(n)\}^{\infty}_{n=1}$ that can be expressed as a perfect square.

7. SMARANDACHE PERMUTATION SEQUENCE $\{PS(n)\}^{\infty}_{n=1}$

For the Smarandache permutation sequence, given in (1.6), the question raised (Dumitrescu and Seleacu [4]) is : *Is there any perfect power among these numbers*?

Smarandache conjectures that there are none. In Theorem 7.1, we prove the conjecture in the affirmative. To prove the theorem, we need the following results.
Lemma 7.1 : For $n \geq 2$, PS(n) is of the form 2(2k+1) for some integer k>1.
Proof : Since for $n \geq 2$,

$$PS(n)=\overline{135\ldots(2n-1)(2n)(2n-2)\ldots42}, \tag{7.1}$$

we see that PS(n) is even and after division by 2, the last digit of the quotient is 1. □

An immediate consequence of the above lemma is the following.
Corollary 7.1 : For $n \geq 2$, $2^\ell \mid PS(n)$ if and only if $\ell=1$.
Theorem 7.1: For $n \geq 1$, PS(n) is not a perfect power.
Proof : The result is clearly true for n=1, since $PS(1)=3\times2^2$ is not a perfect power. The proof for the case $n \geq 2$ is by contradiction.

Let, for some integer $n \geq 2$,

$$PS(n)=x^\ell \text{ for some integers } x>1, \ell \geq 2.$$

Since PS(n) is even, so is x. Let x=2y for some integer y>1. Then,

$$PS(n)=(2y)=2^\ell y^\ell,$$

which shows that $2^\ell \mid PS(n)$, contradicting Corollary 7.1. □

To get more insight into the numbers of the Smaradache permutation sequence, we define a new sequence, called the *reverse even sequence*, and denoted by $\{RES(n)\}^{\infty}_{n=1}$, as follows :

$$RES(n)=\overline{(2n)(2n-2)...42}, \ n\geq 1. \tag{7.2}$$

A first few terms of the sequence are

2, 42, 642, 8642, 108642, 12108642, ...

We note that, for all $n\geq 1$,

$$\begin{aligned} RES(n+1)&=\overline{(2n+2)(2n)(2n-2)...\ 42} \\ &=(2n+2)10^{s}+RES(n) \text{ for some integer } s\geq n, \end{aligned} \tag{7.3}$$

where, more precisely,

s=number of digits in RES(n).

Thus, for example,

$$RES(4)=8\times 10^{3}+RES(3),\ RES(5)=10\times 10^{4}+RES(4),\ RES(6)=12\times 10^{6}+RES(5).$$

Lemma 7.2 : For all $n\geq 1$, $4\,|\,[RES(n+1)-RES(n)]$.

Proof : Since from (7.3),

$$RES(n+1)-RES(n)=(2n+2)10^{s} \text{ for some integer } s\ (\geq n\geq 1),$$

the result follows. □

Lemma 7.3 : The numbers of the reverse even sequence are of the form $2(2k+1)$ for some integer $k\geq 0$.

Proof : The proof is by induction on n. The result is true for n=1. So, we assume that the result is true for some n, that is,

$$RES(n)=2(2k+1) \text{ for some integer } k\geq 0.$$

But, by virtue of Lemma 7.2,

$$RES(n+1)-RES(n)=4k' \text{ for some integer } k'>0,$$

which, together with the induction hypothesis, gives,

$$RES(n+1)=4k'+RES(n)=4(k+k')+2.$$

Thus, the result is true for n+1 as well, completing the proof. □

Lemma 7.4 : $3\,|\,RES(3n)$ if and only if $3\,|\,RES(3n-1)$.

Proof : Since,

$$RES(3n)=(6n)10^{s}+RES(3n-1) \text{ for some integer } s\geq n,$$

the result follows. □

By repeated application of (7.3), we get successively

$$\begin{aligned} RES(n+3)&=(2n+6)10^{s}+RES(n+2) \text{ for some integer } s\geq n+2 \\ &=(2n+6)10^{s}+(2n+4)10^{t}+RES(n+1) \text{ for some integer } t\geq n+1 \\ &=(2n+6)10^{s}+(2n+4)10^{t}+(2n+2)10^{u}+RES(n) \text{ for some integer } u\geq n, \end{aligned} \tag{7.4}$$

so that,

$$RES(n+3)-RES(n)=(2n+6)10^{s}+(2n+4)10^{t}+(2n+2)10^{u}, \tag{7.5}$$

where $s>t>u\geq n\geq 1$.

Lemma 7.5 : $3\,|\,[RES(n+3)-RES(n)]$ for all $n\geq 1$.

Proof : is evident from (7.5), since

$$\begin{aligned} 3\,|\,&(2n+6)10^{s}+(2n+4)10^{t}+(2n+2)10^{u} \\ &=10^{u}[(2n+6)(10^{s-u}+10^{t-u}+1)-2(10^{s-u}+2)]. \ □ \end{aligned}$$

Corollary 7.2 : $3 \mid RES(3n)$ for all $n \geq 1$.

Proof : The result is true for n=1, since RES(3)=642 is divisible by 3. Now, assuming the validity of the result for n, so that $3 \mid RES(3n)$, we get, from Lemma 7.5, $3 \mid RES(3n+3)=RES(3(n+1))$, so that the result is true for n+1 as well.

This completes the proof by induction. □

Corollary 7.3 : $3 \mid RES(3n-1)$ for all $n \geq 1$.

Proof : follows from Lemma 7.4, together with Corollary 7.2. □

Corollary 7.4 : For any $n \geq 0$, $3 \nmid RES(3n+1)$.

Proof : Clearly, the result is true for n=0. For $n \geq 1$, from (7.3),

$$RES(3n+1)=(6n+2)10^s+RES(3n) \text{ for some integer } s \geq 3n.$$

Now, $3 \mid RES(3n)$ (by Corollary 7.2) but $3 \nmid (6n+2)$. Hence the result. □

Using (7.4), we that, for all $n \geq 1$,

$$\begin{aligned}&RES(n+2)-RES(n)\\&=[RES(n+2)-RES(n+1)]+[RES(n+1)-RES(n)]\\&=[(2n+4)10^t-1]RES(n+1)+[(2n+2)10^u-1]RES(n),\end{aligned} \tag{7.6}$$

where t and u are integers with $t>u \geq n+1$.

From (7.6), we get the following result.

Lemma 7.6 : $RES(n+2)-RES(n)>RES(n+1)$ for all $n \geq 1$.

PS(n), given by (7.1), can now be expressed in terms of OS(n) and RES(n) as follows : For any $n \geq 1$,

$$PS(n)=10^s\, OS(n)+RES(n) \text{ for some integer } s \geq n, \tag{7.7}$$

where, more precisely,

s=number of digits in RES(n).

From (7.7), we observe that, for $n \geq 2$, (since $4 \mid 10^s$ for $s \geq n \geq 2$), PS(n) is of the form 4k+2 for some integer k>1, since by Lemma 7.3, RES(n) is of the same form. This provides an alternative proof of Lemma 7.1.

Lemma 7.7 : $3 \mid PS(3n)$ for all $n \geq 1$.

Proof : follows by virtue of Lemma 2.2 and Corollary 7.2, since

$$PS(3n)=10^s\, OS(3n)+RES(3n) \text{ for some integer } s \geq 3n. \text{ □}$$

Lemma 7.8 : $3 \mid PS(n)$ if and only if $3 \mid PS(n+3)$.

Proof : follows by virtue of Lemma 2.1 and Lemma 7.5. □

Lemma 7.9 : $3 \mid PS(3n-2)$ for all $n \geq 1$.

Proof : Since $3 \mid PS(1)=12$, the result is true for n=1. To prove by induction, we assume that the result is true for some n, that is, $3 \mid PS(3n-2)$. But, then, by Lemma 7.8, $3 \mid PS(3n-1)$, showing that the result is true for n+1 as well. □

Lemma 7.10 : For all $n \geq 1$, $PS(n+2)-PS(n)>PS(n+1)$.

Proof : Since

$$PS(n+2)=10^s\, OS(n+2)+RES(n+2) \text{ for some integer } s \geq n+2,$$

$$PS(n+1)=10^t\, OS(n+1)+RES(n+1) \text{ for some integer } t \geq n+1,$$

$$PS(n)=10^u\, OS(n)+RES(n) \text{ for some integer } u \geq n,$$

where s>t>u, we see that

$$\begin{aligned}PS(n+2)-PS(n)&=[10^s\, OS(n+2)-10^u\, OS(n)]+[RES(n+2)-RES(n)]\\&>10^s[OS(n+2)-OS(n)]+[RES(n+2)-RES(n)]\\&>10^t\, OS(n+1)+RES(n+1)=PS(n+1),\end{aligned}$$

where the last inequality follows by virtue of (2.4), Lemma 7.6 and the fact that $10^s>10^t$. □

Lemma 7.10 can be used to prove the following result.

Theorem 7.1 : There are no numbers in the Smarandache permutation sequence that are Fibonacci (or, Lucas) numbers.

Remark 7.1 : The result given in Theorem 7.1 has also been proved by Le [15]. Note that

$$PS(2)=1342=11\times122,\ PS(3)=135642=111\times1222,\ PS(4)=13578642=1111\times12222,$$

as has been pointed out by Zhang [16]. However, such a representation of PS(n) is not valid for $n\geq5$. Thus, the theorem of Zhang [16] holds true only for $1\leq n\leq4$ (and not for $1\leq n\leq9$).

8. SMARANDACHE CONSECUTIVE SEQUENCE $\{CS(n)\}_{n=1}^{\infty}$

The Smarandache consecutive sequence is obtained by repeatedly concatenating the positive integers, and the n-th tem of the sequence is given by (1.7).
Since

$$CS(n+1)=\overline{123\ldots(n-1)n(n+1)},\ n\geq1,$$

we see that, for all $n\geq1$,

$$CS(n+1)=10^s\,CS(n)+(n+1) \text{ for some integer } s\geq1,\ CS(1)=1. \tag{8.1}$$

More precisely,

$$s=\text{number of digits in } (n+1).$$

Thus, for example, $CS(9)=10\,CS(8)+9$, $CS(10)=10^2\,CS(9)+10$.

From (8.1), we get the following result :

Lemma 8.1 : For all $n\geq1$, $CS(n+1)-CS(n)>9\,CS(n)$.

Using Lemma 8.1, we get, following the proof of (2.1),

$$CS(n+2)-CS(n)>9[CS(n+1)+CS(n)] \text{ for all } n\geq1. \tag{8.2}$$

Thus,

$$CS(n+2)-CS(n)>CS(n+1),\ n\geq1. \tag{8.3}$$

Based on computer search for Fibonacci (and Lucas) numbers from 12 up through $CS(2999)=123\ldots29982999$, Asbacher [1] conjectures that (except for the trivial case, $CS(1)=1=F(1)=L(1)$) there are no Fibonacci (and Lucas) numbers in the Smarandache consecutive sequence. The following theorem confirms the conjectures of Ashbacher.

Theorem 8.1 : There are no Fibonacci (and Lucas) numbers in the Smarandache consecutive sequence (except for the trivial cases of $CS(1)=1=F(1)=F(2)=L(1)$, $CS(3)=123=L(10)$).

Proof : is evident from (8.3). □

Remark 8.1 : As has been pointed out by Ashbacher [1], CS(3) is a Lucas number. However, $CS(3)\neq CS(2)+CS(1)$.

Lemma 8.2 : Let $3\mid n$. Then, $3\mid CS(n)$ if and only if $3\mid CS(n-1)$.

Proof : follows readily from (8.1). □

By repeated application of (8.1), we get,

$$\begin{aligned} CS(n+3)&=10^s\,CS(n+2)+(n+3) \text{ for some integer } s\geq1\\ &=10^s[10^t\,CS(n+1)+(n+2)]+(n+3) \text{ for some integer } t\geq1\\ &=10^{s+t}[10^u\,CS(n)+(n+1)]+(n+2)10^s+(n+3) \text{ for some integer } u\geq1\\ &=10^{s+t+u}\,CS(n)+(n+1)10^{s+t}+(n+2)10^s+(n+3), \end{aligned} \tag{8.4}$$

where $s\geq t\geq u\geq1$.

Lemma 8.3 : $3\mid CS(n)$ if and only if $3\mid CS(n+3)$.

Proof : follows from (8.4), since

$$3|[(n+1)10^{s+t}+(n+2)10^s+(n+3)]=[(n+1)(10^{s+t}+10^s+1)+(10^s+2)].\ \square$$

Lemma 8.4 : $3 \mid CS(3n)$ for all $n \geq 1$.

Proof : The proof is by induction on n. The result is clearly true for n=1, since $3 \mid CS(3)=123$. So, we assume that the result is true for some n, that is, we assume that $3 \mid CS(3n)$ for some n. But then, by Lemma 8.4, $3 \mid CS(3n+3)=CS(3(n+1))$, showing that the result is true for n+1 as well, completing induction. □

Corollary 8.1 : $3 \mid CS(3n-1)$ for all $n \geq 1$.

Proof : From (8.1), for $n \geq 1$,

$$CS(3n)=10^s\, CS(3n-1)+(3n) \text{ for some integer } s \geq 1.$$

Since, by Lemma 8.4, $3 \mid CS(3n)$, the result follows. □

Corollary 8.2 : $3 \nmid CS(3n+1)$ for all $n \geq 0$.

Proof : For n=0, CS(1)=1 is not divisible by 3. For $n \geq 1$, from (8.1),

$$CS(3n+1)=10^s\, CS(3n)+(3n+1),$$

where, by Lemma 8.4, $3 \mid CS(3n)$. Since $3 \nmid (3n+1)$, we get desired the result. □

Lemma 8.5 : For any $n \geq 1$, $5 \mid CS(5n)$.

Proof : For $n \geq 1$, from (8.1),

$$CS(5n)=10^s\, CS(5n-1)+(5n) \text{ for some integer } s \geq 1.$$

Clearly, the r.h.s. is divisible by 5. Hence, $5 \mid CS(5n)$. □

For the Smarandache consecutive sequence, the question is : How many terms of the sequence are prime? Fleuren [17] gives a table of prime factors of CS(n) for n=1(1)200, which shows that none of these numbers is prime. In the Editorial Note following the paper of Stephan [18], it is mentioned that, using a supercomputer, no prime has been found in the first 3,072 terms of the Smarandache consecutive sequence. This gives rise to the conjecture that there is no prime in the Smarandache consecutive sequence. This conjecture still remains to be resolved. We note that, in order to check for prime numbers in the Smarandache consecutive sequence, it is sufficient to check the terms of the form CS(3n+1), $n \geq 1$, where 3n+1 is odd and is not divisible by 5.

9. SMARANDACHE REVERSE SEQUENCE $\{RS(n)\}_{n=1}^{\infty}$

The Smarandache reverse sequence is the sequence of numbers formed by concatenating the increasing integers on the left side, starting with RS(1)=1. The n-th term of the sequence is given by (1.8).

Since,

$$RS(n+1)=\overline{(n+1)n(n-1)\ldots 21},\ n \geq 1,$$

we see that, for all, $n \geq 1$,

$$RS(n+1)=(n+1)10^s+RS(n) \text{ for some integer } s \geq n \text{ (with } RS(1)=1) \tag{9.1}$$

More precisely,

$$s=\text{number of digits in } RS(n).$$

Thus, for example,

$$RS(9)=9\times 10^8+RS(8),\ RS(10)=10\times 10^9+RS(9),\ RS(11)=11\times 10^{11}+RS(10).$$

Lemma 9.1 : For all $n \geq 1$, $4 \mid [RS(n+1)-RS(n)]$, $10 \mid [RS(n+1)-RS(n)]$.

Proof : For all $n \geq 1$, from (9.1),

$$RS(n+1)-RS(n)=(n+1)10^s \text{ (with } s \geq n),$$

where the r.h.s. is divisible by both 4 and 10. □

Corollary 9.1 : For all $n\geq 2$, the terms of $\{RS(n)\}_{n=1}^{\infty}$ are of the form $4k+1$.
Proof : The proof is by induction of n. For n=2, the result is clearly true ($RS(2)=21=4\times 5+1$). So, we assume the validity of the result for n, that is, we assume that

$$RS(n)=4k+1 \text{ for integer } k\geq 1.$$

Now, by Lemma 9.1 and the induction hypothesis,

$$RS(n+1)=RS(n)+4k'=4(k+k')+1 \text{ for some integer } k'\geq 1,$$

showing that the result is true for n+1 as well. □
Lemma 9.2 : Let $3 \mid n$ for some n (≥ 2). Then, $3 \mid RS(n)$ if and only if $3 \mid RS(n-1)$.
Proof : follows immediately from (9.1). □

By repeated application of (9.1), we get, for all $n\geq 1$,

$$\begin{aligned} RS(n+3) &= (n+3)10^s+RS(n+2) \text{ for some integer } s\geq n+2 \\ &= (n+3)10^s+(n+2)10^t+RS(n+1) \text{ for some integer } t\geq n+1 \\ &= (n+3)10^s+(n+2)10^t+(n+1)10^u+RS(n) \text{ for some integer } u\geq n, \end{aligned} \tag{9.2}$$

where $s>t>u$. Thus,

$$RS(n+3)=10^u[(n+3)10^{s-u}+(n+2)10^{t-u}+(n+1)]+RS(n). \tag{9.3}$$

Lemma 9.3 : $3 \mid [RS(n+3)-RS(n)]$ for all $n\geq 1$.
Proof : is immediate from (9.3). □

A consequence of Lemma 9.3 is the following.

Corollary 9.2 : $3 \mid RS(3n)$ if and only if $3 \mid RS(n+3)$.

Using Corollary 9.2, the following result can be established by induction on n.

Corollary 9.3 : $3 \mid RS(3n)$ for all $n\geq 1$.
Corollary 9.4 : $3 \mid RS(3n-1)$ for all $n\geq 1$.
Proof : follows from Corollary 9.3, together with Lemma 9.2. □
Lemma 9.4 : $3 \nmid RS(3n+1)$ for all $n\geq 0$.
Proof : The result is true for n=0. For $n\geq 1$, by (9.1),

$$RS(3n+1)=(3n+1)10^s+RS(3n).$$

This gives the desired result, since $3 \mid RS(3n)$ but $3 \nmid (3n+1)$. □

The following result, due to Alexander [19], gives an explicit expression for RS(n) :

Lemma 9.5 : For all $n\geq 1$, $RS(n)=1+\sum_{i=2}^{n} i*10^{\sum_{j=1}^{i-1}(1+\lfloor \log j\rfloor)}$

In Theorem 9.1, we prove that (except for the trivial cases of $RS(1)=1=F(1)=F(2)=L(1)$, $RS(2)=21=F(8)$), the Smarandache reverse sequence contains no Fibonacci and Lucas numbers. For the proof of the theorem, we need the following results.

Lemma 9.6 : For all $n\geq 1$, $RS(n+1)>2RS(n)$.
Proof : Using (9.1), we see that

$$RS(n+1)=(n+1)10^s+RS(n)>2RS(n) \text{ if and only if } RS(n)<(n+1)10^s,$$

which is true since RS(n) is an s-digit number while 10^s is an (s+1)-digit number. □
Corollary 9.5 : For all $n\geq 1$, $RS(n+2)-RS(n)>RS(n+1)$.
Proof : Using (9.2), we have

$$\begin{aligned} RS(n+2)-RS(n) &= [RS(n+2)-RS(n+1)]+[RS(n+1)-RS(n)] \\ &= [(n+2)10^t-(n+1)10^u]+2[RS(n+1)-RS(n)] \\ &> 2[RS(n+1)-RS(n)] \\ &> RS(n+1), \text{ by Lemma 9.6.} \end{aligned}$$

This gives the desired inequality. □

Theorem 9.1: There are no numbers in the Smarandache reverse sequence that are Fibonacci or Lucas numbers (except for the cases of n=1,2).
Proof : follows from Corollary 9.5. □
For the Smarandache reverse sequence, the question is : How many terms of the sequence are prime? By Corollary 9.2 and Corollary 9.3, in searching for primes, it is sufficient to consider the terms of the sequence of the form RS(3n+1), where n>1. In the Editorial Note following the paper of Stephan [18], it is mentioned that searching for prime in the first 2,739 terms of the Smarandache reverse sequence revealed that only RS(82) is prime. This led to the conjecture that RS(82) is the only prime in the Smarandache reverse sequence. However, the conjecture still remains to be resolved. Fleuren [17] presents a table giving prime factors of RS(n) for n=1(1)200, except for the cases n=82,136,139,169.

10. SMARANDACHE SYMMETRIC SEQUENCE $\{SS(n)\}^{\infty}_{n=1}$

The n-th term, SS(n), of the Smarandache symmetric sequence is given by (1.9).
The numbers in the Smarandache symmetric sequence can be expressed in terms of the numbers of the Smarandache consecutive sequence and the Smarandache reverse sequence as follows : For all n≥3,

$$SS(n)=10^s\, CS(n-1)+RS(n-2) \text{ for some integer } s\geq 1, \tag{10.1}$$

with SS(1)=1, SS(2)=11, where more precisely,

s=number of digits in RS(n–2).

Thus, for example, SS(3)=10 CS(2)+RS(1), SS(4)=10^2 CS(3)+RS(2).

Lemma 10.1 : $3\mid SS(3n+1)$ for all n≥1.
Proof : Let n (≥1) be any arbitrary but fixed number. Then, from (10.1),

$$SS(3n+1)=10^s\, CS(3n)+RS(3n-1).$$

Now, by Lemma 8.4, $3\mid CS(3n)$, and by Corollary 9.4, $3\mid RS(3n-1)$. Therefore, $3\mid SS(3n+1)$.
Since n is arbitrary, the lemma is proved. □
Lemma 10.2 : For any n≥1, (1) $3\nmid SS(3n)$, (2) $3\nmid SS(3n+2)$.
Proof : Using (10.1), we see that

$$SS(3n)=10^s\, CS(3n-1)+RS(3n-2),\ n\geq 1.$$

By Corollary 8.1, $3\mid CS(3n-1)$, and by Lemma 9.4, $3\nmid RS(3n-2)$. Hence, CS(3n) cannot be divisible by 3.
Again, since

$$SS(3n+2)=10^s\, CS(3n+1)+RS(3n),\ n\geq 1,$$

and since $3\nmid CS(3n+1)$ (by Corollary 8.2) and $3\mid RS(3n)$ (by Corollary 9.3), it follows that SS(3n+2) is not divisible by 3. □

Using (8.3) and Corollary 9.5, we can prove the following lemma. The proof is similar to that used in proving Lemma 7.10, and is omitted here.
Lemma 10.3 : For all n≥1, SS(n+2)–SS(n)>SS(n+1).
By virtue of the inequality in Lemma 10.3, we have the following.
Theorem 10.1 : (Except for the trivial cases, SS(1)=1=F(1)=L(1), SS(2)=11=L(5)), there are no members of the Smarandache symmetric sequence that are Fibonacci (or, Lucas) numbers.
The following lemma gives the expression of SS(n+1)-SS(n) in terms of CS(n)-CS(n-1).
Lemma 10.4 : SS(n+1)–SS(n)=10^{s+t}[CS(n)–CS(n–2)] for all n≥3, where

s=number of digits in RS(n–2), s+t=number of digits in RS(n–1).

Proof : By (10.1), for $n \geq 3$,

$$SS(n)=10^s\, CS(n-1)+RS(n-2),\ SS(n+1)=10^{s+t}\, CS(n)+RS(n-1),$$

so that

$$\begin{aligned} SS(n+1)-SS(n) &= 10^s[10^t\, CS(n)-CS(n-1)]+[RS(n-1)-RS(n-2)] \\ &= 10^s[10^t CS(n)-CS(n-1)+(n-1)]\ (\text{by}(9.1)). \end{aligned} \qquad (****)$$

But,

$$t=\begin{cases} 1, & \text{if } 2 \leq n-1 \leq 9 \\ m+1, & \text{if } 10^m \leq n-1 \leq 10^{m+1}-1 \text{ (for all } m \geq 1) \end{cases} = \text{number of digits in } (n-1).$$

Therefore, by (8.1)

$$CS(n-1)=10^t\, CS(n-2)+(n-1),$$

and finally, plugging this expression in (****), we get the desired result. □

We observe that SS(2)=11 is prime; the next eight terms of the Smarandache symmetric sequence are composite numbers and squares :

$SS(3)=121=11^2$, $SS(4)=12321=(3\times37)^2=111^2$,
$SS(5)=1234321=(11\times101)^2=1111^2$, $SS(6)=123454321=(41\times271)^2=11111^2$,
$SS(7)=12345654321=(3\times7\times11\times13\times37)^2=111111^2$,
$SS(8)=1234567654321=(239\times4649)^2=1111111^2$,
$SS(9)=123456787654321=(11\times1010101)^2=11111111^2$,
$SS(10)=12345678987654321=(9\times37\times333667)^2=(111\times1001001)^2=111111111^2$.

For the Smarandache symmetric sequence, the question is : How many terms of the sequence are prime? The question still remains to be answered.

11. SMARANDACHE PIERCED CHAIN SEQUENCE $\{PCS(n)\}_{n=1}^{\infty}$

In this section, we give answer to the question posed by Smarandache [5] by showing that, starting from the second term, all the successive terms of the sequence $\{PCS(n)/101\}_{n=1}^{\infty}$, given by (1.11), are composite numbers. This is done in Theorem 11.1 below.

We first observe that the elements of the Smarandache pierced chain sequence, $\{PCS(n)\}_{n=1}^{\infty}$, satisfy the following recurrence relation :

$$PCS(n+1)=10^4\, PCS(n)+101,\ n\geq2;\ PCS(1)=101. \qquad (11.1)$$

Lemma 11.1 : The elements of the sequence $\{PCS(n)\}_{n=1}^{\infty}$ are

$$101,\ 101(10^4+1),\ 101(10^8+10^4+1),\ 101(10^{12}+10^8+10^4+1),\ \ldots,$$

and in general,

$$PCS(n)=101[10^{4(n-1)}+10^{4(n-2)}+\ldots+10^4+1],\ n\geq1. \qquad (11.2)$$

Proof : The proof of (11.2) is by induction on n. The result is clearly true for n=1. So, we assume that the result is true for some n.

Now, from (11.1) together with the induction hypothesis, we see that

$$\begin{aligned} PCS(n+1) &= 10^4\, PCS(n)+101 \\ &= 10^4[101(10^{4(n-1)}+10^{4(n-2)}+\ldots+10^4+1)]+101 \\ &= 101(10^{4n}+10^{4(n-1)}+\ldots+10^4+1), \end{aligned}$$

which shows that the result is true for n+1. □

It has been mentioned in Ashbacher [1] that PCS(n) is divisible by 101 for all $n\geq1$, and Lemma 11.1 shows that this is indeed the case. Another consequence of Lemma 11.1 is the following corollary.

Corollary 11.1 : The elements of the sequence $\{PCS(n)/101\}_{n=1}^{\infty}$ are

$$1,\ x+1,\ x^2+x+1,\ x^3+x^2+x+1,\ \ldots,$$

and in general,

$$PCS(n)/101=x^{n-1}+x^{n-2}+\ldots+1,\ n\geq 1, \qquad (11.3)$$

where $x\equiv 10^4$.

Theorem 11.1 : For all $n\geq 2$, $PCS(n)/101$ is a composite number.

Proof : The result is true for n=2. In fact, the result is true if n is even as shown below : If n (≥ 4) is even, let n=2m for some integer m (≥ 2). Then, from (11.3),

$$\begin{aligned} PCS(2m)/101 &= x^{2m-1}+x^{2m-2}+\ldots+x+1 \\ &= x^{2m-2}(x+1)+\ldots+(x+1) \\ &= (x+1)(x^{2m-2}+x^{2m-4}+\ldots+1) \end{aligned}$$

that is, $$PCS(2m)/101=(10^4+1)[10^{8(m-1)}+10^{8(m-2)}+\ldots+1], \qquad (11.4)$$

which shows that PCS(2m)/101 is a composite number for all m (≥ 2).

Next, we consider the case when n is prime, say n=p, where p (≥ 3) is a prime. In this case, from (11.3),

$$PCS(p)/101=x^{p-1}+x^{p-2}+\ldots+1=(x^p-1)/(x-1).$$

Let $y=10^2$ (so that $x=y^2$). Then,

$$\frac{PCS(p)}{101}=\frac{x^p-1}{x-1}=\frac{y^{2p}-1}{y^2-1}=\frac{(y^p-1)(y^p+1)}{(y+1)(y-1)}$$

$$=\frac{\{(y-1)(y^{p-1}+y^{p-2}+\ldots+1)\}\{(y+1)(y^{p-1}-y^{p-2}+\ldots+1)\}}{(y+1)(y-1)}$$

$$=(y^{p-1}-y^{p-2}+y^{p-3}-\ldots+1)(y^{p-1}+y^{p-2}+y^{p-3}+\ldots+1)$$

that is, $$PCS(p)/101=[10^{2(p-1)}-10^{2(p-2)}+10^{2(p-3)}+\ldots+1][10^{2(p-1)}+10^{2(p-2)}+\ldots+1], \qquad (11.5)$$

so that SPC(p)/101 is a composite number for each prime p (≥ 3).

Finally, we consider the case when n is odd but composite. Then, letting n=pr where p is the largest prime factor of n and r (≥ 2) is an integer, we see that

$$\begin{aligned} PCS(n)/101 &= PCS(pr)/101 \\ &= x^{pr-1}+x^{pr-2}+\ldots+1 \\ &= x^{p(r-1)}(x^{p-1}+x^{p-2}+\ldots+1)+x^{p(r-2)}(x^{p-1}+x^{p-2}+\ldots+1)+\ldots \\ &\quad +(x^{p-1}+x^{p-2}+\ldots+1) \\ &= (x^{p-1}+x^{p-2}+\ldots+1)[x^{p(r-1)}+x^{p(r-2)}+\ldots+1] \end{aligned}$$

that is, $$PCS(n)/101=[10^{4(p-1)}+10^{4(p-2)}+\ldots+1][10^{4p(r-1)}+10^{4p(r-2)}+\ldots+1], \qquad (11.6)$$

and hence, PCS(n)/101=PCS(pr)/101 is also a composite number.

All these complete the proof of the theorem. □

Remark 11.1 : The Smarandache pierced chain sequence has been studied by Le [20] and Kashihara [21] as well. Following different approaches, they have proved by contradiction that for $n\geq 2$, PCS(n)/101 is not prime. In Theorem 11.1, we have proved the same result by actually finding out the factors of PCS(n)/101 for all $n\geq 2$. Kashihara [21] raises the question : Is the sequence PCS(n)/101 square-free for $n\geq 2$? From (11.4), (11.5) and (11.6), we see that the answer to the question of Kashihara is yes.

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