SOME REMARKS CONCERNING THE DISTRIBUTION OF THE SMARANDACHE FUNCTION

by

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The Smarandache function is a numerical function $S: \mathbb{N}^* \rightarrow \mathbb{N}^*$ $S(k)$ representing the smallest natural number $n$ such that $n!$ is divisible by $k$. From the definition it results that $S(1)=1$.

I will refer for the beginning the following problem:

"Let $k$ be a rational number, $0 < k \leq 1$. Does the diophantine equation $\frac{S(n)}{n} = k$ has always solutions? Find all $k$ such that the equation has an infinite number of solutions in $\mathbb{N}^*$" from "Smarandache Function Journal".

I intend to prove that equation hasn't always solutions and case that there are an infinite number of solutions is when $k = \frac{1}{r}, r \in \mathbb{N}^*, k \in \mathbb{Q}$ and $0 < k \leq 1 \Rightarrow$ there are two relatively prime non negative integers $p$ and $q$ such that $k = \frac{p}{q}, p,q \in \mathbb{N}^*, 0 < q \leq p$. Let $n$ be a solution of the equation $\frac{S(n)}{n} = k$. Then $\frac{S(n)}{n} = \frac{p}{q}$, (1). Let $d$ be a highest common divisor of $n$ and $S(n)$ : $d = (n, S(n))$. The fact that $p$ and $q$ are relatively prime and (1) implies that $S(n) = qd$ , $n = pd \Rightarrow S(pd) = qd$ (*).

This equality gives us the following result: $(qd)!$ is divisible by $pd \Rightarrow [(qd - 1)! - q]$ is divisible by $p$. But $p$ and $q$ are relatively prime integers, so $(qd-1)!$ is divisible by $p$. Then $S(p) \leq qd - 1$.

I prove that $S(p) \geq (q - 1)d$.

If we suppose against all reason that $S(p) < (q - 1)d$, it means $[(q - 1)d - 1]$ is divisible by $p$. Then $(pd) | [(q - 1)d]$! because $d \mid (q - 1)d$, so $S(pd) \leq (q - 1)d$. This is contradiction with the fact that $S(pd) = qd > (q - 1)d$. We have the following inequalities:

$(q - 1)d \leq S(p) \leq qd - 1$.

For $q \geq 2$ we have from the first inequality $d \leq \frac{S(p)}{q - 1}$ and from the second $\frac{S(p+1)}{q} \leq d$, so

$\frac{S(p+1)}{q} \leq d \leq \frac{S(p)}{q - 1}$.
For \( k = \frac{q}{p} \), \( q \geq 2 \), the equations has solutions if and only if there is a natural number between \( \frac{S(p+1)}{q} \) and \( \frac{S(p)}{q-1} \). If there isn't such a number, then the equation hasn't solutions. However, if there is a number \( d \) with \( \frac{S(p+1)}{q} \leq d \leq \frac{S(p)}{q-1} \), this doesn't mean that the equation has solutions. This condition is necessary but not sufficient for the equation to have solutions.

For example:

a) \( k = \frac{4}{5} \), \( q = 4 \), \( p = 5 \) \( \Rightarrow \frac{S(p+1)}{q} = \frac{6}{4} = \frac{3}{2} \), \( \frac{S(p)}{q-1} = \frac{5}{3} \). In this case the equation hasn't solutions.

b) \( k = \frac{3}{10} \), \( q = 3 \), \( p = 10 \) ; \( S(10) = 5 \), \( \frac{6}{3} = 2 \leq d \leq \frac{5}{2} \). If the equation has solutions, then we must have \( d = 2 \), \( n = dp = 20 \), \( S(n) = dq = 6 \). But \( S(20) = 5 \).

This is a contradiction. So there are no solutions for \( k = \frac{3}{10} \).

We can have more than natural numbers between \( \frac{S(p+1)}{q} \) and \( \frac{S(p)}{q-1} \). For example:

\( k = \frac{3}{29} \), \( q = 3 \), \( p = 29 \), \( \frac{S(p+1)}{q} = 10 \), \( \frac{S(p)}{q-1} = 14.5 \).

We prove that the equation \( \frac{S(n)}{n} = k \) hasn't always solutions.

If \( q \geq 2 \) then the number of solutions is equal with the number of values of \( d \) that verify relation (*). But \( d \) can be a nonnegative integer between \( \frac{S(p+1)}{q} \) and \( \frac{S(p)}{q-1} \), so \( d \) can take only a finite set of values. This means that the equation has no solutions or it has only a finite number of solutions.

We study now case \( k = \frac{1}{p} \), \( p \in \mathbb{N}^* \). In this case the equation has an infinite number of solutions. Let \( p_0 \) be a prime number such that \( p < p_0 \) and \( n = pp_0 \). We have \( S(n) = S(pp_0) = p \), so \( S(n) = p_0 \), \( \frac{S(n)}{n} = \frac{p_0}{pp_0} = \frac{1}{p} \), so the equation has an infinite number of solutions.

I will refer now to another problem concerning the ratio \( \frac{S(n)}{n} \) "Is there an infinity of natural numbers such that \( 0 < \left\{ \frac{x}{S(x)} \right\} < \left\{ \frac{S(x)}{x} \right\} ?" from the same journal.

I will prove that the only number \( x \) that verifies the inequalities is \( x = 9 \) : \( S(9) = 6 \), \( \frac{S(x)}{x} = \frac{6}{9} = \frac{2}{3} \), \( \frac{x}{S(x)} = \frac{9}{6} = \frac{3}{2} \) and \( 0 < \frac{1}{2} < \frac{2}{3} \), so \( x = 9 \) verifies \( 0 < \left\{ \frac{x}{S(x)} \right\} < \left\{ \frac{S(x)}{x} \right\} \).

Let \( x = p_1^{a_1} \ldots p_n^{a_n} \) be the standard form of \( x \).

\( S(x) = \max_{1 \leq k \leq n} S(p_k^{a_k}) \). We put \( S(x) = S(p^a) \), where \( p^a \) is one of \( p_1^{a_1} \ldots p_n^{a_n} \) such that
\[ S(p^a) = \max_{1 \leq k \leq n} S(p_k^{a_k}). \]
can take one of the following values: $\frac{1}{S(x)}$, $\frac{2}{S(x)}$, $\ldots$, $\frac{S(x)-1}{S(x)}$ because $0 < \left< \frac{x}{S(x)} \right> < \left< \frac{S(x)}{x} \right>$ (We have $S(x) \leq x$, so $\frac{S(x)}{x} \leq 1$ and $\left< \frac{S(x)}{x} \right> \leq S(x)$). This means $S(x) \geq \frac{1}{S(x)} \Rightarrow S(p^2)^{2} > x \geq p^a$. (2)

But $(ap)! = 1 \cdot 2 \cdot \ldots \cdot (p-1) \cdot 2 \cdot \ldots \cdot (2p)$, $(ap)$ is divisible by $p^a$, so $ap \geq S(p^a)$. From this last inequality and (2) it follows that $\alpha^2 p^2 > p^2$. We have three cases:

I. $\alpha = 1$. In this case $S(x) = S(p) = p$, $x$ is divisible by $p$, so $\frac{x}{p} \in \mathbb{Z}$. This is a contradiction.

There are no solutions for $\alpha = 1$.

II. $\alpha = 2$. In this case $S(x) = S(p^2) = 2p$, because $p$ is a prime number and $(2p)! = 1 \cdot 2 \cdot \ldots \cdot (p-1) \cdot 2 \cdot \ldots \cdot (2p)$, so $S(p^2) = 2p$.

But $\left< \frac{px}{2} \right> \in \left< 0, \frac{1}{2} \right>$. This means $\left< \frac{px}{2} \right> = \frac{1}{2} \Rightarrow \frac{1}{2} < \frac{2}{px} < 4$; $p$ is a prime number $\Rightarrow p \in \{2,3\}$.

If $p = 2$ and $px_1 < 4 \Rightarrow x_1 = 1$, but $x = 4$ isn’t a solution of the equation: $S(4) = 4$ and $\left< \frac{4}{4} \right> = 0$.

If $p = 3$ and $px_1 < 4 \Rightarrow x_1 = 1$, so $x = 3^2 = 9$ is a solution of the equation.

III. $\alpha = 3$. We have $\alpha^2 p^2 > p^a \Rightarrow \alpha^2 > p^{a-1}$.

For $\alpha \geq 8$ we prove that we have $p^{a-2} > p^2$, $(\forall) p \in \mathbb{N}^*$, $p \geq 2$.

We prove by induction that $2^{n-1} > (n-1)^2$.

$2^{n-1} = 2 \cdot 2^{n-2} \geq n \cdot 2^{n-2} \geq n^2 \geq 8n \geq n^2 + 2n + 1 = (n-1)^2$, because $n \geq 8$.

We proved that $p^{a-2} \geq 2\alpha-1 \geq 2\alpha^2$, for any $\alpha \geq 8$, $p \in \mathbb{N}^*$, $p \geq 2$.

We have to study the case $\alpha \in \{3,4,5,6,7\}$.

a) $\alpha = 3 \Rightarrow p \in \{2,3,5,7\}$, because $p$ is a prime number.

If $p = 2$ then $S(x) = S(2^3) = 4$. But $x$ is divisible by $8$, so $\left< \frac{x}{S(x)} \right> = \left< \frac{x}{4} \right> = 0$, so $x = 4$ cannot be a solution of the inequality.

If $p = 3 \Rightarrow S(x) = S(3^3) = 9$. But $x$ is divisible by $3$, so $\left< \frac{x}{S(x)} \right> = \left< \frac{x}{9} \right> = 0$, so $x = 9$ cannot be a solution of the inequality.

If $p = 5 \Rightarrow S(x) = S(5^3) = 15$; $\left< \frac{x}{S(x)} \right> = \left< \frac{S(x)}{x} \right> = 0$ $x = 5^3 \cdot x_1$, $x_1 \in \mathbb{N}^*$, $(5,x_1) = 1$.48
We have \( 0 < \left\lfloor \frac{5 \cdot x}{3} \right\rfloor < \left\lfloor \frac{3}{5 \cdot x_1} \right\rfloor \). This first inequality implies \( \left\lfloor \frac{5 \cdot x}{3} \right\rfloor \in \left\{ \frac{1}{3}, \frac{2}{3} \right\} \), so \( \frac{1}{3} < \frac{3}{5 \cdot x_1} \Rightarrow 5 \cdot x_1 < 9 \), but this is impossible.

If \( p=7 \Rightarrow S(x)=S(7^3)=21, x=7^3 \cdot x_1 \), \((7, x_1)=1, x_1 \in \mathbb{N}^*\).

We have \( 0 < \left\lfloor \frac{x}{S(x)} \right\rfloor < \left\lfloor \frac{S(x)}{x} \right\rfloor \Rightarrow 0 < \left\lfloor \frac{7^2 \cdot x}{3} \right\rfloor < \frac{3}{7^2 \cdot x_1} \). But \( 0 < \left\lfloor \frac{7^2 \cdot x}{3} \right\rfloor \) implies \( \left\lfloor \frac{7^2 \cdot x}{3} \right\rfloor \in \left\{ \frac{1}{3}, \frac{2}{3}, \frac{3}{3} \right\} \).

W have \( \frac{1}{3} \leq \left\lfloor \frac{7^2 \cdot x}{3} \right\rfloor \Rightarrow 7^2 \cdot x_1 < 9 \), but is impossible.

b) \( \alpha=4 : 16 \Rightarrow p \in \{2,3\} \).

If \( p=2 \Rightarrow S(x)=S(x^2)=6, x=16 \cdot x_1 \), \(x_1 \in \mathbb{N}^*, (2, x_1)=1\), \(0 < \left\lfloor \frac{x}{S(x)} \right\rfloor < \frac{S(x)}{x} \Rightarrow 0 < \left\lfloor \frac{16}{S(x)} \right\rfloor = \frac{16}{6} = \frac{2}{3} \cdot 3 \cdot \frac{3}{8} \), so the inequality isn’t verified.

If \( p=3 \Rightarrow S(x)=S(3^4)=9, x=3^4 \cdot x_1 \), \((3, x_1)=1 \Rightarrow 9|x \Rightarrow \frac{x}{S(x)} = 0 \), so the inequality isn’t verified.

For \( \alpha=\{5,6,7\} \), the only natural number \( p>1 \) that verifies the inequality \( \alpha^2 > p^{\alpha-2} \) is 2:
\( \alpha=5 : 25 > p^3 \Rightarrow p=2 \)
\( \alpha=6 : 36 > p^4 \Rightarrow p=2 \)
\( \alpha=7 : 49 > p \)

In every case \( x=2^\alpha \cdot x_1 \), \(x_1 \in \mathbb{N}^*, (x_1, 2)=1\), and \(S(x_1) \leq S(2^\alpha)\).

But \( S(2^5) = S(2^6) = S(2^7) = 8 \), so \( S(x) = 8 \). But \( x \) is divisible by 8, so \( \left\lfloor \frac{x}{S(x)} \right\rfloor = 0 \) so the inequality isn’t verified because \( 0 < \left\lfloor \frac{x}{S(x)} \right\rfloor \). We found that there is only \( x=9 \) to verify the inequality \( 0 < \left\lfloor \frac{x}{S(x)} \right\rfloor < \left\lfloor \frac{S(x)}{x} \right\rfloor \).

I try to study some diophantine equations proposed in "Smarandache Function Journal".

1) I study the equation \( S(mx)=mS(x) \), \( m \geq 2 \) and \( x \) is a natural number.
Let $x$ be a solution of the equation.

We have $S(x)!$ is divisible by $x$. It is known that among $m$ consecutive numbers, one is divisible by $m$, so $(S(x)!)$ is divisible by $m$, so $(S(x)+1)(S(x)+2)...(S(x)+m)$ is divisible by $(mx)$. We know that $S(mx)$ is the smallest natural number such that $S(mx)!$ is divisible by $(mx)$ and this implies $S(mx) \leq S(x)^m$. But $S(mx) = mS(x)$, so $mS(x) \leq S(x)^m \iff mS(x) - S(x)^m - 1 \leq (m-1)(S(x)-1).$ We have several cases:

If $m=1$ then the equation becomes $S(x) = S(x)$, so any natural number is a solution of the equation.

If $m=2$, we have $S(x) \in \{1, 2\}$ implies $x \in \{1, 2\}$. We conclude that if $m=1$ then any natural number is a solution of the equation; if $m=2$ then $x=1$ and $x=2$ are only solution and if $m \geq 3$ the only solution of the equation is $x=1$.

2) Another equation is $S(xy) = xy$, $x, y$ are natural numbers.

Let $(x, y)$ be a solution of the equation.

$(yx)! = 1 \cdot (x+1) \cdot (2x) \cdot ... \cdot (yx)$ implies $S(xy) \leq yx$, so $y \leq yx_1$ because $S(xy) = yx$.

But $y \geq 1$, so $yx \geq x$.

If $x=1$ then equation becomes $S(1) = y$, so $y=1$, so $x=y=1$ is a solution of the equation.

If $x \geq 2$ then $x \geq 2^{x-1}$. But the only natural numbers that verify this inequality are $x=y=2$: $x=y=2$ verifies the equation, so $x=y=2$ is a solution of the equation.

For $x \geq 3$ we prove that $x < 2^{x-1}$. We make the proof by induction.

If $x=3$ : $3 < 2^{3-1} = 4$.

We suppose that $k < 2^{k-1}$ and we prove that $k-1 < 2^k$. We have $2^k = 2 \cdot 2 \cdot 2 \cdot ... \cdot 2 = k + k + k + 1$, so the inequality is established and there are no other solutions then $x=y=1$ and $x=y=2$.

3) I will prove that for any $m, n$ natural numbers, if $m>1$ then the equation $S(x^n) = x^m$ has no solution or it has a finite number of solutions, and for $m=1$ the equation has a infinite number of solutions.

I prove that $S(x^n) \leq nx$. But $x^m = S(x^n)$, so $x^m \leq nx$.

For $m \geq 2$ we have $x^{m-1} \leq n$. If $m=2$ then $x \leq n$, and if $m \geq 3$ then $x \leq n^{1/m}$, so $x$ can take only a finite number of values, so the equation can have only a finite number of solutions or it has no solutions.

We notice that $x=1$ is a solution of the equation for any $m, n$ natural numbers.
If the equation has a solution different from 1, we must have $x^m = S(x^n) \leq x^n$, so $m \leq n$.

If $m=n$, the equation becomes $x^{m+n} = S(x^n)$, so $x^n$ is a prime number or $x^n = 4$, so $n=1$ and any prime number as well as $x=4$ is a solution of the equation, or $n=2$ and the only solutions are $x=1$ and $x=2$.

For $m=1$ and $n \geq 1$, we prove that the equations $S(x^m) = x^m$, $x \in N^*$ has an infinite number of solutions. Let be a prime number, $p > n$. We prove that $(np)$ is a solution of the equation, that is $S((np)^n) = np$.

$n < p$ and $p$ is a prime number, so $n$ and $p$ are relatively prime numbers.

$n < p$ implies:

$(np)! = 1 \cdot 2 \cdot \ldots \cdot n(n+1) \cdot \ldots \cdot (2n) \cdot \ldots \cdot (pn)$ is divisible by $n^a$.

$(np)! = 1 \cdot 2 \cdot \ldots \cdot p(p+1) \cdot \ldots \cdot (2p) \cdot \ldots \cdot (pn)$ is divisible by $p^n$.

But $p$ and $n$ are relatively prime numbers, so $(np)!$ is divisible by $(np)^n$.

If we suppose that $S((np)^n) < np$, then we find that $(np-1)!$ is a divisible by $(np)^n$, so $(np-1)!$ is divisible by $p^n(3)$. But the exponent of $p$ in the standard form of $p$ in the standard form of $(np-1)!$ is:

$$E = \left\lfloor \frac{np-1}{p} \right\rfloor + \left\lfloor \frac{np-1}{p^2} \right\rfloor + \ldots$$

But $p > n$, so $p^2 > np > np-1$. This implies:

$$\left\lfloor \frac{np-1}{p^k} \right\rfloor = 0 \quad \text{for any } k \geq 2.$$ We have:

$$E = \left\lfloor \frac{np-1}{p} \right\rfloor = n-1.$$

This means $(np-1)!$ is divisible by $p^{n-1}$, but isn’t divisible by $p^n$, so this is a contradiction with (3). We proved that $S((np)^n) = np$, so the equation $S(x^n) = x$ has an infinite number of solutions for any natural number $n$.

REFERENCE


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