REMARKS ON SOME OF THE SMARANDACHE'S PROBLEMS. Part 1

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In 1996 the author of this remarks wrote reviews for "Zentralblatt für Mathematik" for books [1] and [2] and this was his first contact with the Smarandache's problems. He solved some of them and he published his solutions in [3]. The present paper contains some of the results from [3].

In [1] Florentin Smarandache formulated 105 unsolved problems, while in [2] C. Dumitrescu and V. Seleacu formulated 140 unsolved problems of his. The second book contains almost all the problems from [1], but now each problem has unique number and by this reason the author will use the numeration of the problems from [2]. Also, in [2] there are some problems, which are not included in [1].

When the text of [3] was ready, the author received Charles Ashbacher's book [4] and he corrected a part of the prepared results having in mind [4].

We shall use the usual notations: $[x]$ and $\lceil x \rceil$ for the integer part of the real number x and for the least integer $\geq x$, respectively.

The 4-th problem from [2] (see also 18-th problem from [1]) is the following:
Smarandache's deconstructive sequence:

$$\underbrace{1,\ \ 23,\ \ 456,\ \ 789}\underbrace{1,\ \ 23456,\ \ 789}\underbrace{123,\ \ 456789}\underbrace{1,\ \ 23456789},\ \ \underbrace{123456789},\ \ \ldots$$

Let the n-th term of the above sequence be a_n. Then we can see that the first digits of the first nine members are, respectively: 1, 2, 4, 7, 2, 7, 4, 2, 1. Let us define the function ω as follows:

r	0	1	2	3	4	5	6	7	8	9
$\omega(r)$	1	1	2	4	7	2	7	4	2	1

Here we shall use the arithmetic function ψ, discussed shortly in the Appendix and detailed in the author's paper [5] .

In [3] it is proved that the form of the $n-$th member of the above sequence is

$$a_n = \overline{b_1 b_2 \ldots b_n},$$

where

$$b_1 = \omega(n - [\frac{n}{9}])$$

$$b_2 = \psi(\omega(n - [\frac{n}{9}]) + 1)$$

$$\ldots$$

$$b_n = \psi(\omega(n - [\frac{n}{9}]) + n - 1).$$

To the above sequence $\{a_n\}_{n=1}^{\infty}$ we can juxtapose the sequence $\{\psi(a_n)\}_{n=1}^{\infty}$ for which we can prove (as above) that its basis is $[1,5,6,7,2,3,4,8,9]$.

The problem can be generalized, e.g., to the following form:

Study the sequence $\{a_n\}_{n=1}^{\infty}$, *with its* $s-$*th member of the form*

$$a_s = \overline{b_1 b_2 \ldots b_{s.k}},$$

where $b_1 b_2 \ldots b_{s.k} \in \{1, 2, \ldots, 9\}$ *and*

$$b_1 = \omega'(s - [\frac{s}{9}])$$

$$b_2 = \psi(\omega'(s - [\frac{s}{9}]) + 1)$$

$$\ldots$$

$$b_{s.k} = \psi(\omega'(s - [\frac{s}{9}]) + s.k - 1),$$

and here

$$\begin{array}{c|cccccc} r & 1 & 2 & 3 & 4 & 5 & 6 \\ \omega(r) & 1 & \psi(k+1) & \psi(3k+1) & \psi(6k+1) & \psi(10k+1) & \psi(15k+1) \end{array}$$

$$\begin{array}{c|ccc} r & 7 & 8 & 9 \\ \omega(r) & \psi(21k+1) & \psi(28k+1) & \psi(36k+1) \end{array}$$

To the last sequence $\{a_n\}_{n=1}^{\infty}$ we can juxtapose again the sequence $\{\psi(a_n)\}_{n=1}^{\infty}$ for which we can prove (as above) that its basis is $[3,9,3,6,3,6,9,8,9]$.

The 16-th problem from [2] (see also 21-st problem from [1]) is the following:

Digital sum:

$$\underbrace{0,1,2,3,4,5,6,7,8,9}, \underbrace{1,2,3,4,5,6,7,8,9,10}, \underbrace{2,3,4,5,6,7,8,9,10,11},$$

$$\underbrace{3,4,5,6,7,8,9,10,11,12}, \underbrace{4,5,6,7,8,9,10,11,12,13}, \underbrace{5,6,7,8,9,10,11,12,13,14}, \ldots$$

($d_s(n)$ is the sum of digits.) Study this sequence.

The form of the general term a_n of the sequence is:

$$a_n = n - 9. \sum_{k=1}^{\infty} [\frac{n}{10^k}].$$

It is not always true that equality $d_s(m) + d_s(n) = d_s(m+n)$ is valid. For example,

$$d_s(2) + d_s(3) = 2 + 3 = 5 = d_s(5),$$

but

$$d_s(52) + d_s(53) = 7 + 8 = 15 \neq 6 = d_s(105).$$

The following assertion is true

$$d_s(m+n) = \begin{cases} d_s(m) + d_s(n), \text{ if } d_s(m) + d_s(n) \leq 9.\max([\frac{d_s(m)}{9}], [\frac{d_s(n)}{9}]) \\ \\ d_s(m) + d_s(n) - 9.\max([\frac{d_s(m)}{9}], [\frac{d_s(n)}{9}]), \text{ otherwise} \end{cases}$$

The sum of the first n members of the sequence is

$$S_n = 5.[\frac{n}{10}].([\frac{n}{10}] + 8) + (n - 10.[\frac{n}{10}]).(\frac{n-1}{2} - 4.[\frac{n}{10}]).$$

The 37-th and 38-th problems from [2] (see also 39-th problem from [1]) are the following:

(Inferior) prime part:

2, 3, 3, 5, 5, 7, 7, 7, 7, 11, 11, 13, 13, 13, 13, 17, 17, 19, 19, 19, 19, 23, 23, 23, 23, 23, 23, 29, 29, 31,

31, 31, 31, 31, 31, 37, 37, 37, 37, 41, 41, 43, 43, 43, 43, 47, 47, 47, 47, 47, 47, 53, 53, 53, 53, 53, ...

(For any positive real number n one defines $p_p(n)$ as the largest prime number less than or equal to n.)

(Superior) prime part:

$$2, 2, 2, 3, 5, 5, 7, 7, 11, 11, 11, 11, 13, 13, 17, 17, 17, 17, 19, 19, 23, 23, 23, 23, 29, 29, 29, 29, 29, 29,$$

$$31, 31, 37, 37, 37, 37, 37, 37, 41, 41, 41, 41, 43, 43, 47, 47, 47, 47, 53, 53, 53, 53, 53, 53, 59, 59, \ldots$$

(For any positive real number n one defines $P_p(n)$ as the smallest prime number greater than or equal to n.)

Study these sequences.

First, we should note that in the first sequence $n \geq 2$, while in the second one $n \geq 0$. It would be better, if the first two members of the second sequence are omitted. Let everywhere below $n \geq 2$.

Second, let us denote by

$$\{p_1, p_2, p_3, \ldots\} = \{2, 3, 5, \ldots\},$$

the set of all prime numbers. Let $p_0 = 1$, and let $\pi(n)$ be the number of the prime numbers less or equal to n.

Then the n–th member of the first sequence is

$$p_p(n) = p_{\pi(n)-1}$$

and of the second sequence is

$$P_p(n) = p_{\pi(n)+\mathcal{B}(n)},$$

where

$$\mathcal{B}(n) = \begin{cases} 0, & \text{if } n \text{ is a prime number} \\ 1, & \text{otherwise} \end{cases}$$

(see [7]).

The checks of these equalities are straightforward, or by induction.

Therefore, the values of the n-th partial sums of the two sequences are, respectively,

$$X_n = \sum_{k=1}^{n} p_p(k) = \sum_{k=2}^{\pi(n)} (p_k - p_{k-1}).p_{k-1} + (n - p_{\pi(n)} + 1).p_{\pi(n)}$$

$$Y_n = \sum_{k=1}^{n} P_p(k) = \sum_{k=1}^{\pi(n)} (p_k - p_{k-1}).p_k + (n - p_{\pi(n)}).p_{\pi(n)+B(n)}.$$

The 39-th and 40-th problems from [2] (see also 40-th problem from [1]) are the following:
(Inferior) square part:

$$0, 1, 1, 1, 4, 4, 4, 4, 4, 9, 9, 9, 9, 9, 9, 9, 16, 16, 16, 16, 16, 16, 16, 16, 16, 25, 25, 25, 25, 25, 25,$$

$$25, 25, 25, 25, 25, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 49, 49, \ldots$$

(the largest square less than or equal to n.)
(Superior) square part:

$$0, 1, 4, 4, 4, 9, 9, 9, 9, 9, 16, 16, 16, 16, 16, 16, 16, 25, 25, 25, 25, 25, 25,$$

$$25, 25, 25, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 49, 49, \ldots$$

(the smallest square greater than or equal to n.) Study these sequences.

The 41-st and 42-nd problems from [1] (see also 41-st problem from [1]) are the following:
(Inferior) cube part:

$$0, 1, 1, 1, 1, 1, 1, 1, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 27, 27, 27, 27, 27, 27, 27, 27, 27, \ldots$$

(the largest cube less than or equal to n.)
(Superior) cube part:

$$0, 1, 8, 8, 8, 8, 8, 8, 8, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 27, 64, \ldots$$

(the smallest cube greater than or equal to n.) Study these sequences.

The n-th term of each of the above sequences is, respectively

$$a_n = [\sqrt{n}]^2, \quad b_n = \lceil \sqrt{n} \rceil^2, \quad c_n = [\sqrt[3]{n}]^3, \quad d_n = \lceil \sqrt[3]{n} \rceil^3.$$

The values of the n-th partial sums of these sequences are:

$$A_n = \frac{[\sqrt{n} - 1]([\sqrt{n} - 1] + 1)(3[\sqrt{n} - 1]^2 + 5[\sqrt{n} - 1] + 1)}{6} + (n - [\sqrt{n}]^2 + 1).[\sqrt{n}]^2,$$

$$B_n = \frac{[\sqrt{n}]([\sqrt{n}]+1)(3[\sqrt{n}]^2+[\sqrt{n}]-1)}{6} + (n-[\sqrt{n}]^2).\lceil\sqrt{n}\rceil^2,$$

$$C_n = \frac{[\sqrt[3]{n}-1]([\sqrt[3]{n}-1]+1)(5[\sqrt[3]{n}-1]^4+16[\sqrt[3]{n}-1]^3}{10}$$

$$+\frac{14[\sqrt[3]{n}-1]^2+[\sqrt[3]{n}-1]-1)}{10} + (n-[\sqrt[3]{n}]^3+1).[\sqrt[3]{n}]^3,$$

$$D_n = \frac{[\sqrt[3]{n}]([\sqrt[3]{n}]+1)(5[\sqrt[3]{n}]^4+4[\sqrt[3]{n}]^3-4[\sqrt[3]{n}]^2-[\sqrt[3]{n}]+1)}{10}(n-[\sqrt[3]{n}]^3+1).\lceil\sqrt[3]{n}\rceil^3.$$

The 43-rd and 44-th problems from [2] (see also 42-nd problem from [1]) are the following:
(Inferior) factorial part:

$$1, 2, 2, 2, 2, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, \ldots$$

($F_p(n)$ is the largest factorial less than or equal to n.)
(Superior) factorial part:

$$1, 2, 6, 6, 6, 6, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 120, \ldots$$

($f_p(n)$ is the smallest factorial greater than or equal to n.) Study these sequences.

It must be noted immediately that p is not an index in F_p and f_p.

First, we shall extend the definition of the function "factorial" (possibly, it is already defined, but the author does not know this). It is defined only for natural numbers and for a given such number n it has the form:

$$n! = 1.2. \ldots .n.$$

Let the new form of the function "factorial" be the following for the real positive number y:

$$y! = y.(y-1).(y-2)\ldots(y-[y]+1),$$

where $[y]$ denotes the integer part of y.

Therefore, for the real number $y > 0$:

$$(y+1)! = y!.(y+1).$$

This new factorial has Γ−representation

$$y! = \frac{\Gamma(y+1)}{\Gamma(y-[y]+1)}$$

and representation by the Pochhammer symbol

$$y! = (y)_{[y]}$$

(see, e.g., [8]).

Obviously, if y is a natural number, $y!$ is the standard function "factorial".

It can be easily seen that the extended function has the properties similar to these of the standard function.

Second, we shall define a new function (possibly, it is already defined, too, but the author does not know this). It is an inverse function of the function "factorial" and for the arbitraty positive real numbers x and y it has the form:

$$x? = y \ \underline{\text{iff}} \ y! = x.$$

Let us show only one of its integer properties.

For every positive real number x:

$$[(x+1)?] = \begin{cases} [x?]+1, & \text{if there exists a natural number } n \text{ such} \\ & \text{that } \ n! = x+1 \\ & \\ [x?], & \text{otherwise} \end{cases}$$

From the above discussion it is clear that we can ignore the new factorial, using the definition

$$x? = y \ \underline{\text{iff}} \ (y)_{[y]} = x.$$

Practically, everywhere below y is a natural number, but at some places x will be a positive real number (but not an integer).

Then the n−th member of the first sequence is

$$F_p(n) = [n?]!$$

and of the second sequence it is

$$f_p(n) = \lceil n?\rceil !.$$

The checks of these equalities is direct, or by the method of induction.

Therefore, the values of the n-th partial sums of the two above Smarandache's sequences are, respectively,

$$X_n = \sum_{k=1}^{n} F_p(k) = \sum_{k=1}^{[n?]} (k! - (k-1)!).(k-1)! + (n - [n?]! + 1).[n?]!$$

$$Y_n = \sum_{k=1}^{n} f_p(k) = \sum_{k=1}^{[n?]} (k! - (k-1)!).k! + (n - [n?]! + 1).\lceil n?\rceil !$$

The 100-th problem from [2] (see also 80-th problem from [1]) is the following:

Square roots:

$$0, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 4, 5, 5, 5, 5, 5, 5, \ldots$$

($s_q(n)$ is the superior integer part of square root of n.)

Remark: this sequence is the natural sequence, where each number is repeated $2n+1$ times, because between n^2 (included) and $(n+1)^2$ (excluded) there are $(n+1)^2 - n^2$ different numbers.

Study this sequence.

The 101-st problem from [2] (see also 81-st problem from [1]) is the following:

Cubical roots:

$$0, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, \ldots$$

($c_q(n)$ is the superior integer part of cubical root of n.)

Remark: this sequence is the natural sequence, where each number is repeated $3n^2 + 3n + 1$ times, because between n^3 (included) and $(n+1)^3$ (excluded) there are $(n+1)^3 - n^3$ different numbers.

Study this sequence.

The 102-nd problem from [2] (see also 82-nd problem from [1]) is the following:

$m-$power roots:

($m_q(n)$ is the superior integer part of m-power root of n.)

Remark: this sequence is the natural sequence, where each number is repeated $(n+1)^m - n^m$ times.

Study this sequence.

The n-th term of each of the above sequences is, respectively,

$$x_n = [\sqrt{n}], \quad y_n = [\sqrt[3]{n}], \quad z_n = [\sqrt[m]{n}]$$

and the values of the n-th partial sums are, respectively,

$$X_n = \sum_{k=1}^{n} x_k, = \frac{([\sqrt{n}]-1)[\sqrt{n}](4[\sqrt{n}]+1)}{6} + n - [\sqrt{n}]^2 + 1).[\sqrt{n}],$$

$$Y_n = \sum_{k=1}^{n} y_k = \frac{([\sqrt[3]{n}-1)[\sqrt[3]{n}]^2(3[\sqrt[3]{n}+1)}{4} + (n - [\sqrt[3]{n}]^3 + 1).[\sqrt[3]{n}],$$

$$Z_n = \sum_{k=1}^{n} z_k = \sum_{k=1}^{n} (([\sqrt[m]{k}]+1)^m - [\sqrt[m]{k}]^m)[\sqrt[m]{k}-1]^m + (n - [\sqrt[m]{n}]^m + 1).[\sqrt[m]{n}].$$

The 118-th Smarandache's problem (see [2]) is:

"Smarandache's criterion for coprimes":

If a, b are strictly positive integers, then: a and b are coprimes if and only if

$$a^{\varphi(b)+1} + b^{\varphi(a)+1} \equiv a + b (\text{mod } ab),$$

where φ is Euler's totient.

For the natural number

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where $p_1, p_2, \ldots, p_k$ are different prime numbers and $\alpha_1, \alpha_2, \ldots, \alpha_k \geq 1$ are natural numbers, the Euler's totient is defined by:

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i - 1).$$

Below we shall introduce a solution of one direction of this problem and we shall introduce a counterexample to the other direction of the problem.

Let a, b are strictly positive integers for which $(a, b) = 1$. Hence, from one of the Euler's theorems:

If m and n are natural numbers and $(m, n) = 1$, then

$$m^{\varphi(n)} \equiv 1 (\text{mod } n)$$

(see, e.g., [6]) it follows that

$$a^{\varphi(b)} \equiv 1 (\text{mod } b)$$

and

$$b^{\varphi(b)} \equiv 1 (\text{mod } a).$$

Therefore,

$$a^{\varphi(b)+1} \equiv a (\text{mod } ab)$$

and

$$b^{\varphi(b)+1} \equiv b (\text{mod } ab)$$

from where it follows that really

$$a^{\varphi(b)+1} + b^{\varphi(a)+1} \equiv a + b (\text{mod } ab).$$

It can be easily seen that the other direction of the Smarandache's problem is not valid. For example, if $a = 6$ and $b = 10$, and, therefore, $(a, b) = 2$, then:

$$6^{\varphi(10)+1} + 10^{\varphi(6)+1} = 6^5 + 10^3 = 7776 + 1000 = 8776 \equiv 16 (\text{mod } 60).$$

Therefore, the "Smarandache's criterion for coprimes" is valid only in the form:

If a, b are strictly positive coprime integers, then

$$a^{\varphi(b)+1} + b^{\varphi(a)+1} \equiv a + b (\text{mod } ab).$$

The 125-th Smarandache's problem (see [2]) is the following:

To prove that

$$n! > k^{n-k+1} \prod_{i=0}^{k-1} [\frac{n-i}{k}]! \qquad (*)$$

for any non-null positive integers n and k.

Below we shall introduce a solution to the problem.

First, let us define for every negative integer m : $m! = 0$.

Let everywhere k be a fixed natural number. Obviously, if for some n: $k > n$, then the inequality (*) is obvious, because its right side is equal to 0. Also, it can be easily seen that (*) is valid for $n = 1$. Let us assume that (*) is valid for some natural number n. Then,

$$(n+1)! - k^{n-k+2} \prod_{i=0}^{k-1} [\frac{n-i+1}{k}]!$$

(by the induction assumption)

$$> (n+1).k^{n-k+1} \prod_{i=0}^{k-1} [\frac{n-i}{k}]! - k^{n-k+2} \prod_{i=0}^{k-1} [\frac{n-i+1}{k}]!$$

$$= k^{n-k+1} \prod_{i=0}^{k-2} [\frac{n-i}{k}]!.((n+1).[\frac{n-k+1}{k}]! - k.[\frac{n+1}{k}]!) \geq 0,$$

because

$$(n+1).[\frac{n-k+1}{k}]! - k.[\frac{n+1}{k}]!$$

$$= (n+1).[\frac{n-k+1}{k}]! - k.[\frac{n-k+1}{k} + 1]!$$

$$= [\frac{n-k+1}{k}]!.(n+1-k.[\frac{n+1}{k}]) \geq 0.$$

Thus the problem is solved.

Finally, we shall formulate two new problems:

1. Let $y > 0$ be a real number and let k be a natural number. Will the inequality

$$y! > k^{y-k+1} \prod_{i=0}^{k-1} [\frac{y-i}{k}]!$$

be valid again?

2. For the same y and k will the inequality

$$y! > k^{y-k+1} \prod_{i=0}^{k-1} \frac{y-i}{k}!$$

be valid?

The paper and the book [3] are based on the author's papes [9-16].

APPENDIX

Here we shall describe two arithmetic functions which were used below, following [5].

For

$$n = \sum_{i=1}^{m} a_i.10^{m-i} \equiv \overline{a_1 a_2 ... a_m},$$

where a_i is a natural number and $0 \leq a_i \leq 9$ $(1 \leq i \leq m)$ let (see [5]):

$$\varphi(n) = \begin{cases} 0 & , \text{ if } n = 0 \\ \sum_{i=1}^{m} a_i & , \text{ otherwise} \end{cases}$$

and for the sequence of functions $\varphi_0, \varphi_1, \varphi_2, ...$, where ($l$ is a natural number)

$$\varphi_0(n) = n,$$

$$\varphi_{l+1} = \varphi(\varphi_l(n)),$$

let the function ψ be defined by

$$\psi(n) = \varphi_l(n),$$

in which

$$\varphi_{l+1}(n) = \varphi_l(n).$$

This function has the following (and other) properties (see [5]):

$$\psi(m+n) = \psi(\psi(m) + \psi(n)),$$

$$\psi(m.n) = \psi(\psi(m).\psi(n)) = \psi(m.\psi(n)) = \psi(\psi(m).n),$$

$$\psi(m^n) = \psi(\psi(m)^n),$$

$$\psi(n+9) = \psi(n),$$

$$\psi(9n) = 9.$$

Let the sequence $a_1, a_2, ...$ with members - natural numbers, be given and let

$$c_i = \psi(a_i) \ (i = 1, 2, ...).$$

Hence, we deduce the sequence $c_1, c_2, ...$ from the former sequence. If k and l exist, so that $l \geq 0$,

$$c_{i+l} = c_{k+i+l} = c_{2k+i+l} = ...$$

for $1 \leq i \leq k$, then we shall say that

$$[c_{l+1}, c_{l+2}, ..., c_{l+k}]$$

is a base of the sequence $c_1, c_2, ...$ with a length of k and with respect to function ψ.

For example, the Fibonacci sequence $\{F_i\}_{i=0}^{\infty}$, for which

$$F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n \ (n \geq 0)$$

has a base with a length of 24 with respect to the function ψ and it is the following:

$$[1, 1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 9, 8, 8, 7, 6, 4, 1, 5, 6, 2, 8, 1, 9];$$

the Lucas sequence $\{L_i\}_{i=0}^{\infty}$, for which

$$L_2 = 2, L_1 = 1, L_{n+2} = L_{n+1} + L_n \ (n \geq 0)$$

also has a base with a length of 24 with respect to the function ψ and it is the following:

$$[2,1,3,4,7,2,9,2,2,4,6,1,7,8,6,5,2,7,9,7,7,5,3,8];$$

even the Lucas-Lehmer sequence $\{l_i\}_{i=0}^{\infty}$, for which

$$l_1 = 4, l_{n+1} = l_n^2 - 2 \ (n \geq 0)$$

has a base with a length of 1 with respect to the function ψ and it is [5].

The $k-th$ triangular number t_k is defined by the formula

$$t_k = \frac{k(k+1)}{2}$$

and it has a base with a length of 9 with the form

$$[1,3,6,1,5,3,1,9,9].$$

It is directly checked that the bases of the sequences $\{n^k\}_{k=1}^{\infty}$ for $n = 1,2,...,9$ are the ones introduced in the following table.

n	a base of a sequence $\{n^k\}_{i=1}^{\infty}$	a length of the base
1	1	1
2	2,4,8,7,5,1	6
3	9	1
4	4,7,1	3
5	5,7,8,4,2,1	6
6	9	1
7	7,4,1	3
8	8,1	2
9	9	1

On the other hand, the sequence $\{n^n\}_{n=1}^{\infty}$ has a base (with a length of 9) with the form

$$[1,4,9,1,2,9,7,1,9],$$

and the sequence $\{k^{n!}\}_{n=1}^{\infty}$ has a base with a length of 9 with the form

$$\begin{cases} [1] & \text{, if } k \neq 3m \text{ for some natural number } m \\ [9] & \text{, if } k = 3m \text{ for some natural number } m \end{cases}$$

We must note that in [5] there are some misprints, corrected here.

An obvious, but unpublished up to now result is that the sequence $\{\psi(n!)\}_{n=1}^{\infty}$ has a base with a length of 1 with respect to the function ψ and it is [9]. The first members of this sequence are

$$1, 2, 6, 6, 3, 9, 9, 9, ...$$

We shall finish with two new results related to the concept "factorial" which occur in some places in this book.

The concepts of $n!!$ is already introduced and there are some problems in [1,2] related to it. Let us define the new factorial $n!!!$ only for numbers with the forms $3k + 1$ and $3k + 2$:

$$n!!! = 1.2.4.5.7.8.10.11...n$$

We shall prove that the sequence $\{\psi(n!!!)\}_{n=1}^{\infty}$ has a base with a length of 12 with respect to the function ψ and it is

$$[1, 2, 8, 4, 1, 8, 8, 7, 1, 5, 8, 1].$$

Really, the validity of the assertion for the first 12 natural numbers with the above mentioned forms, i.e., the numbers

$$1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17,$$

is directly checked. Let us assume that the assertion is valid for the numbers

$$(18k + 1)!!!, (18k + 2)!!!, (18k + 4)!!!, (18k + 5)!!!, (18k + 7)!!!, (18k + 8)!!!,$$

$$(18k + 10)!!!, (18k + 11)!!!, (18k + 13)!!!, (18k + 14)!!!, (18k + 16)!!!,$$

$$(18k + 17)!!!.$$

Then

$$\psi((18k + 19)!!!) = \psi((18k + 17)!!!.(18k + 19))$$

$$= \psi(\psi(18k + 17)!!!.\psi(18k + 19))$$

$$= \psi(1.1) = 1;$$

$$\psi((18k + 20)!!!) = \psi((18k + 19)!!!.(18k + 20))$$

$$= \psi(\psi(18k + 19)!!!.\psi(18k + 20))$$

$$= \psi(1.2) = 2;$$

$$\psi((18k+22)!!!) = \psi((18k+20)!!!.(18k+22))$$

$$= \psi(\psi(18k+20)!!!.\psi(18k+22))$$

$$= \psi(2.4) = 8,$$

etc., with which the assertion is proved.

Having in mind that every natural number has exactly one of the forms $3k+3$, $3k+1$ and $3k+2$, for the natural number $n = 3k+m$, where $m \in \{1,2,3\}$ and $k \geq 1$ is a natural number, we can define:

$$n!_m = \begin{cases} 1.4...(3k+1), & \text{if } n = 3k+1 \text{ and } m = 1 \\ \\ 2.5...(3k+2), & \text{if } n = 3k+2 \text{ and } m = 2 \\ \\ 3.6...(3k+3), & \text{if } n = 3k+3 \text{ and } m = 3 \end{cases}$$

As above, we can prove that:

- for the natural number n with the form $3k+1$, the sequence $\{\psi(n!_1)\}_{n=1}^{\infty}$ has a base with a length of 3 with respect to the function ψ and it is

$$[1, \psi(3k+1), 1];$$

- for the natural number n with the form $3k+2$, the sequence $\{\psi(n!_1)\}_{n=1}^{\infty}$ has a base with a length of 6 with respect to the function ψ and it is

$$[2, \psi(6k+4), 8, 7, \psi(3k+5), 1];$$

- for the natural number n with the form $3k+3$, the sequence $\{\psi(n!_1)\}_{n=1}^{\infty}$ has a base with a length of 1 with respect to the function ψ and it is [9] and only its first member is 3.

Now we can see that

$$n!!! = \begin{cases} (3k+1)!_1.(3k-1)!_2, & \text{if } n = 3k+1 \text{ and } k \geq 1 \\ \\ (3k+1)!_1.(3k+2)!_2, & \text{if } n = 3k+2 \text{ and } k \geq 1 \end{cases}$$

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