A result obtained using Smarandache Function

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Smarandache Function is defined as followed:
$S(m) =$ The smallest positive integer so that $S(m)!$ is divisible by $m$. [1]
Let's see the value which such function takes for $m = p^n$ with $n$ integer, $n \geq 2$
and $p$ prime number. To do so a Lemma is required.

Lemma 1 $\forall \ m, n \in \mathbb{N} \ m, n \geq 2$

$m^n = E \left[ \frac{m^{n+1} - m^n + m}{m} \right] + E \left[ \frac{m^{n+1} - m^n + m}{m^2} \right] + \cdots + E \left[ \frac{m^{n+1} - m^n + m}{m^{E[\log_m(m^{n+1} - m^n + m)]}} \right]$

Where $E(x)$ gives the greatest integer less than or equal to $x$.

Demonstration:

Let's see in the first place the value taken by $E[\log_m(m^{n+1} - m^n + m)]$.
If $n \geq 2$: $m^{n+1} - m^n + m < m^{n+1}$ and therefore $\log_m(m^{n+1} - m^n + m) < \log_m m^{n+1} = n + 1$. As a result $E[\log_m(m^{n+1} - m^n + m)] < n + 1$.
And if $m \geq 2$: $mn^n \geq 2m^n \Rightarrow m^{n+1} \geq 2m^n \Rightarrow m^{n+1} + m \geq 2m^n \Rightarrow m^{n+1} - m^n + m \geq m^n \Rightarrow \log_m(m^{n+1} - m^n + m) \geq \log_m m^n = n \Rightarrow E[\log_m(m^{n+1} - m^n + m)] \geq n$
As a result: $n \leq E[\log_m(m^{n+1} - m^n + m)] < n + 1$ therefore:
$E[\log_m(m^{n+1} - m^n + m)] = n$ if $n, m \geq 2$

Now let's see the value which it takes for $1 \leq k \leq n$: $E \left[ \frac{m^{n+1} - m^n + m}{m^k} \right]$

$E \left[ \frac{m^{n+1} - m^n + m}{m^k} \right] = E \left[ m^{n+1-k} - m^{n-k} + \frac{1}{m^{k-1}} \right]$

If $k = 1$: $E \left[ \frac{m^{n+1} - m^n + m}{m} \right] = m^n - m^{n-1} + 1$
If $1 < k \leq n$: $E \left[ \frac{m^{n+1} - m^n + m}{m^k} \right] = m^{n+1-k} - m^{n-k}$
Let's see what is the value of the sum:

\[
\begin{align*}
  k = 1 & : m^n - m^{n-1} - m^{n-2} - m^{n-3} + 1 \\
  k = 2 & : m^{n-1} - m^{n-2} \\
  k = 3 & : m^{n-2} - m^{n-3} \\
  \vdots & \quad \vdots \\
  k = n - 1 & : m^2 - m \\
  k = n & : m - 1
\end{align*}
\]

Therefore:

\[
\sum_{k=1}^{n} E \left[ \frac{m^{n+1} - m^n + m}{m^k} \right] = m^n \quad m, n \geq 2
\]

Proposition 1 \(\forall p\) prime number \(\forall n \geq 2\):

\[
S(p^n) = p^{n+1} - p^n + p
\]

Demonstration:

Having \(e_p(k)\) = exponent of the prime number \(p\) in the prime numbers decomposition of \(k\).

We get:

\[
e_p(k!) = E\left(\frac{k}{p}\right) + E\left(\frac{k}{p^2}\right) + E\left(\frac{k}{p^3}\right) + \cdots + E\left(\frac{k}{p^{E[\log_p k]}}\right)
\]

And using the Lemma we have:

\[
e_p[(p^{n+1} - p^n + p)!] = E\left(\frac{p^{n+1} - p^n + p}{p}\right) + E\left(\frac{p^{n+1} - p^n + p}{p^2}\right) + \cdots + E\left(\frac{p^{n+1} - p^n + p}{p^{E[\log_p (p^{n+1} - p^n + p)]}}\right) = p^n
\]

Therefore:

\[
\frac{(p^{n+1} - p^n + p)!}{p^{p^n}} \in \mathbb{N} \quad \text{and} \quad \frac{(p^{n+1} - p^n + p - 1)!}{p^{p^n}} \notin \mathbb{N}
\]

And:

\[
S(p^n) = p^{n+1} - p^n + p
\]

References:


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