MORE RESULTS AND APPLICATIONS OF THE GENERALIZED SMARANDACHE STAR FUNCTION

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ABSTRACT: In [1] we define SMARANDACHE FACTOR PARTITION FUNCTION, as follows:

Let \( \alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r \) be a set of \( r \) natural numbers and \( p_1, p_2, p_3, \ldots p_r \) be arbitrarily chosen distinct primes then \( F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) \) called the Smarandache Factor Partition of \((\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r)\) is defined as the number of ways in which the number

\[
N = \frac{\alpha_1 \alpha_2 \alpha_3 \ldots \alpha_r}{p_1 p_2 p_3 \ldots p_r}
\]

could be expressed as the product of its divisors. For simplicity, we denote \( F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) = F'(N), \) where

\[
N = \frac{\alpha_1 \alpha_2 \alpha_3 \ldots \alpha_r \alpha_{n}}{p_1 p_2 p_3 \ldots p_r \ldots p_n}
\]

and \( p_r \) is the \( r^{th} \) prime. \( p_1 = 2, p_2 = 3 \) etc.

Also for the case

\[
\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_r = \ldots = \alpha_n = 1
\]

Let us denote

\[
F(1, 1, 1, 1, 1, \ldots) = F(1\#n)
\]

\[
\leftarrow n \text{- ones} \rightarrow
\]

Function as follows:

Smarandache Star Function

(1) \( F'(N) = \sum_{d|r, d|N} F'(d) \) where \( d_r | N \)

(2) \( F^{**}(N) = \sum_{d_r/N} F^{**}(d) \)

d_r ranges over all the divisors of \( N \).

If \( N \) is a square free number with \( n \) prime factors, let us denote

\[
F^{**}(N) = F^{**}(1#n)
\]

Smarandache Generalised Star Function

(3) \( F^{n*}(N) = \sum_{d_r/N} F^{(n-1)*}(d) \)

\( n > 1 \)

and \( d_r \) ranges over all the divisors of \( N \).

For simplicity we denote

\[
F'(Np_1p_2...p_n) = F'(N@1#n), \text{ where} \quad (N,p_i) = 1 \text{ for } i = 1 \text{ to } n \text{ and each } p_i \text{ is a prime.}
\]

\( F'(N@1#n) \) is nothing but the Smarandache factor partition of (a number \( N \) multiplied by \( n \) primes which are coprime to \( N \)).

In [3] I had derived a general result on the Smarandache Generalised Star Function. In the present note some more results and applications of Smarandache Generalised Star Function are explored and derived.
DISCUSSION:

THEOREM (4.1):

\[ F'_{r*}(p^\alpha) = \sum_{k=0}^{\alpha} r^{k-1} C_{r-1} P(\alpha-k) \quad -----(4.1) \]

Following proposition shall be applied in the proof of this

\[ \sum_{k=0}^{\alpha} r^{k-1} C_{r-1} = \alpha^r C_r \quad -----(4.2) \]

Let the proposition (4.1) be true for \( n = r \) to \( n = 1 \).

\[ F'_{r*}(p^\alpha) = \sum_{k=0}^{\alpha} r^{k-1} C_{r-1} P(\alpha-k) \quad -----(4.3) \]

\[ F'_{r*(r+1)*}(p^\alpha) = \sum_{t=0}^{\alpha} F'_{r*}(p^t) \]

( \( p \) ranges over all the divisors of \( p^\alpha \) for \( t = 0 \) to \( \alpha \) )

RHS = \( F'_{r*}(p^\alpha) + F'_{r*}(p^{\alpha-1}) + F'_{r*}(p^{\alpha-2}) + \ldots + F'_{r*}(p^1) + F'_{r*}(1) \)

from the proposition (4.3) we have

\[ F'_{r*}(p^\alpha) = \sum_{k=0}^{\alpha} r^{k-1} C_{r-1} P(\alpha-k) \]

expanding RHS from \( k = 0 \) to \( \alpha \)

\[ F'_{r*}(p^\alpha) = r^{\alpha-1} C_{r-1} P(0) + r^{\alpha-2} C_{r-1} P(1) + \ldots + r^{-1} C_{r-1} P(\alpha) \]

similarly

\[ F'_{r*}(p^{\alpha-1}) = r^{\alpha-2} C_{r-1} P(0) + r^{\alpha-3} C_{r-1} P(1) + \ldots + r^{-1} C_{r-1} P(\alpha-1) \]

\[ F'_{r*}(p^{\alpha-2}) = r^{\alpha-3} C_{r-1} P(0) + r^{\alpha-4} C_{r-1} P(1) + \ldots + r^{-1} C_{r-1} P(\alpha-2) \]

\[ \ldots \]

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\[ F^{(r+1)}(p) = C_{r-1} P(0) + C_{r-1} P(1) \]
\[ F^{(r+1)}(1) = C_{r-1} P(0) \]

Summing up left and right sides separately we find that the
\[ \text{LHS} = F^{(r+1)}(p^a) \]

The RHS contains \( \alpha + 1 \) terms in which \( P(0) \) occurs, \( \alpha \) terms in which \( P(1) \) occurs etc.

\[ \text{RHS} = \sum_{k=0}^{\alpha} C_{r-1} P(0) + \sum_{k=0}^{\alpha-1} C_{r-1} P(1) + \ldots + \sum_{k=0}^{1} C_{r-1} P(\alpha-1) \]

\[ + \sum_{k=0}^{\alpha} C_{r-1} P(\alpha) \]

Applying proposition (4.2) to each of the \( \sum \) we get
\[ \text{RHS} = C_r P(0) + C_r P(1) + C_r P(2) + \ldots + C_r P(\alpha) \]
\[ = \sum_{k=0}^{\alpha} r^k C_r P(\alpha-k) \]

Or
\[ F^{(r+1)}(p^a) = \sum_{k=0}^{\alpha} r^k C_r P(\alpha-k) \]

The proposition is true for \( n = r+1 \), as we have
\[ F^{(r)}(p^a) = \sum_{k=0}^{\alpha} P(\alpha-k) = \sum_{k=0}^{\alpha} C_0 P(\alpha-k) = \sum_{k=0}^{\alpha} C_{k-1} P(\alpha-k) \]

The proposition is true for \( n = 1 \)

Hence by induction the proposition is true for all \( n \).

This completes the proof of theorem (4.1).

Following theorem shall be applied in the proof of theorem (4.3)

THEOREM (4.2)

\[ \frac{n-r}{256} \]
\[ \sum_{k=0}^{n-r} \binom{n}{r+k} r^k \binom{r}{k} m^k = \binom{n}{r} (1+m)^{(n-r)} \]

**PROOF:**

LHS = \[ \sum_{k=0}^{n-r} \binom{n}{r+k} r^k \binom{r}{k} m^k \]

\[ = \sum_{k=0}^{n-r} \frac{(n)!}{((r+k)!(n-r-k)!)} \cdot \frac{(r+k)!}{(k)!(r)!} \cdot m^k \]

\[ = \sum_{k=0}^{n-r} \frac{(n)!}{((r)!(n-r)!)} \cdot \frac{(n-r)!}{((k)!(n-r-k)!)} \cdot m^k \]

\[ = \binom{n}{r} \sum_{k=0}^{n-r} \binom{n-r}{k} m^k \]

\[ = \binom{n}{r} (1+m)^{(n-r)} \]

This completes the proof of theorem (4.2)

**THEOREM (4.3):**

\[ F^{m\ast}(1\#n) = \sum_{r=0}^{n} \frac{n}{r} \binom{n}{r} m^{n-r} F(1\#r) \]

**Proof:**

From theorem (2.4) (ref.[1]) we have

\[ F\ast(1\#n) = F(1\#(n+1)) = \sum_{r=0}^{n} \frac{n}{r} \binom{n}{r} F(1\#r) = \sum_{r=0}^{n} \frac{n}{r} \binom{n}{r} (1)^{n-r} F(1\#r) \]

hence the proposition is true for \( m = 1 \).

Let the proposition be true for \( m = s \). Then we have

\[ F^{s\ast}(1\#n) = \sum_{r=0}^{n} \frac{n}{r} \binom{n}{r} s^{n-r} F(1\#r) \]

or

\[ F^{s\ast}(1\#0) = \sum_{r=0}^{0} \frac{n}{r} \binom{n}{r} s^{0-r} F(1\#0) \]

\[ F^{s\ast}(1\#1) = \sum_{r=0}^{1} \frac{n}{r} \binom{n}{r} s^{1-r} F(1\#1) \]

\[ F^{s\ast}(1\#2) = \sum_{r=0}^{2} \frac{n}{r} \binom{n}{r} s^{2-r} F(1\#1) \]

\[ F^{s\ast}(1\#3) = \sum_{r=0}^{3} \frac{n}{r} \binom{n}{r} s^{3-r} F(1\#3) \]

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\[
F^s(1#0) = 0^C_0 F(1#0) \quad -----(0)
\]

\[
F^s(1#1) = 1^C_0 F^1(1#0) + 1^C_1 F^0(1#1) \quad -----(1)
\]

\[
F^s(1#2) = 2^C_0 F^2(1#0) + 2^C_1 F^1(1#1) + 2^C_2 F^0(1#2) \quad -----(2)
\]

\[
F^s(1#r) = r^C_0 F^r(1#0) + r^C_1 F^1(1#1) + \ldots + r^C_r F^0(1#r) \quad -----(r)
\]

\[
F^s(1#n) = n^C_0 F^0(1#0) + n^C_1 F^1(1#1) + \ldots + n^C_n F^0(1#n) \quad -----(n)
\]

Multiplying the \(r\)th equation with \(n^C_r\) and then summing up we get

the RHS as

\[
= [n^C_0^0 C_0 s^0 + n^C_1^1 C_0 s^1 + n^C_2^2 C_0 s^2 + \ldots + n^C_k^k C_0 s^k + \ldots + n^C_n^n C_0 s^n]F(1#0)
\]

\[
[n^C_1^1 C_1 s^0 + n^C_2^2 C_1 s^1 + n^C_3^3 C_1 s^2 + \ldots + n^C_k^k C_1 s^k + \ldots + n^C_n^n C_1 s^n]F(1#1)
\]

\[
[n^C_r^r C_r s^0 + n^C_{r+1}^{r+1} C_r s^1 + \ldots + n^C_{r+k}^{r+k} C_r s^k + \ldots + n^C_n^n C_r s^n]F(1#r)
\]

\[
+ n^C_n^n C_n s^n]F(1#n)
\]

\[
= \sum_{r=0}^{n} \left\{ \sum_{k=0}^{n-r} n^C_{r+k} r^k C_r s^k \right\} F(1#r)
\]

\[
= \sum_{r=0}^{n} n^C_r (1+s)^{n-r} F(1#n) \quad , \text{by theorem (4.2)}
\]

LHS \(= \sum_{r=0}^{n} n^C_r F^s(1#r)\)

Let \(N = p_1 p_2 p_3 \ldots p_n\). Then there are \(n^C_r\) divisors of \(N\) containing

exactly \(r\) primes. Then LHS = the sum of the \(s^{th}\) Smarandache

star functions of all the divisors of \(N\). \(= F'(s+1)^*(N) = F'(s+1)^*(1#n)\).

Hence we have

\[
F'(s+1)^*(1#n) = \sum_{r=0}^{n} n^C_r (1+s)^{n-r} F(1#n)
\]
\( F^{s\ast}(1\#0) = 0^\circ C \times F(1\#0) \)  \hspace{1cm} (0) \\
\( F^{s\ast}(1\#1) = 1^\circ C \times s^1 \times F(1\#0) + 1^\circ C \times s^0 \times F(1\#1) \)  \hspace{1cm} (1) \\
\( F^{s\ast}(1\#2) = 2^\circ C \times s^2 \times F(1\#0) + 2^\circ C \times s^1 \times F(1\#1) + 2^\circ C \times s^0 \times F(1\#2) \)  \hspace{1cm} (2) \\
\[ 
\vdots
\] \\
\( F^{s\ast}(1\#r) = r^\circ C \times s^r \times F(1\#0) + r^\circ C \times s^1 \times F(1\#1) + \ldots + r^\circ C \times s^0 \times F(1\#r) \)  \hspace{1cm} (r) \\
\[ 
\vdots
\] \\
\( F^{s\ast}(1\#n) = n^\circ C \times s^n \times F(1\#0) + n^\circ C \times s^1 \times F(1\#1) + \ldots + n^\circ C \times s^0 \times F(1\#n) \)  \hspace{1cm} (n) \\

multiplied the \( r \)th equation with \( n^\circ C \times s^r \) and then summing up we get the RHS as

\[ 
= [n^\circ C \times s^0 \times F(1\#0) + n^\circ C \times s^1 \times F(1\#1) + \ldots + n^\circ C \times s^n \times F(1\#n)] \\
\] 

\[ 
\sum_{r=0}^{n} \sum_{k=0}^{n-r} n^\circ C \times s^r \times F(1\#r) \\
= \sum_{r=0}^{n} n^\circ C \times (1+s)^{n-r} \times F(1\#n) \\
\] 

by theorem (4.2) 

LHS = \( \sum_{r=0}^{n} n^\circ C \times F^{s\ast}(1\#r) \) 

Let \( N = p_1 p_2 p_3 \ldots p_n \). Then there are \( n^\circ C \times s^r \) divisors of \( N \) containing exactly \( r \) primes. Then LHS = the sum of the \( s^\ast \) Smarandache star functions of all the divisors of \( N \). = \( F'(s^\ast+1)\ast(N) = F(s^\ast+1)\ast(1\#n) \). 

Hence we have 

\[ 
F(s^\ast+1)\ast(1\#n) = \sum_{r=0}^{n} n^\circ C \times (1+s)^{n-r} \times F(1\#n) \\
\]
which takes the same format
\[ P(s) \Rightarrow P(s+1) \]

and it has been verified that the proposition is true for \( m = 1 \)
hence by induction the proposition is true for all \( m \).
\[
F^{m*}(1\#n) = \sum_{r=0}^{n} \binom{n}{r} m^{n-r} F(1\#r)
\]

This completes the proof of theorem (4.3)

NOTE:
From theorem (3.1) we have
\[
F'(N@1\#n) = F'(Np_1p_2 \ldots p_n) = \sum_{m=0}^{n} a_{(n,m)} F^{m*}(N)
\]
where
\[
a_{(n,m)} = \frac{1}{m!} \sum_{k=1}^{m} (-1)^{m-k} \cdot \binom{m}{k} \cdot k^n
\]

If \( N = p_1p_2 \ldots p_k \) Then we get
\[
F(1\#(k+n)) = \sum_{m=0}^{n} \left[ a_{(n,m)} \sum_{t=0}^{k} \binom{k}{t} \cdot m^{k-t} \cdot F(1\#t) \right]_{m=0, t=0}^{k} \quad -------(4.4)
\]
The above result provides us with a formula to express \( B_n \) in terms of smaller Bell numbers. It is in a way generalisation of theorem (2.4) in Ref [5].

THEOREM(4.4):
\[
F(\alpha,1\#(n+1)) = \sum_{k=0}^{\alpha} \sum_{r=0}^{n} \binom{n}{r} F(k,1\#r)
\]

PROOF: LHS = \( F(\alpha,1\#(n+1)) = F'(p^{\alpha} p_1p_2p_3 \ldots p_{n+1}) = F'*(p^{\alpha} p_1p_2p_3 \ldots p_n) + \sum F' (\text{all the divisors containing only } p^0) + \sum F' (\text{all the
divisors containing only $p^1) + \sum F'$ (all the divisors containing only
$p^2) + \ldots + \sum F'$ (all the divisors containing only $p^r)
= \sum_{r=0}^{n} \binom{n}{r} F(0, 1#r) + \sum_{r=0}^{n} \binom{n}{r} F(1, 1#r) + \sum_{r=0}^{n} \binom{n}{r} F(2, 1#r) + \sum_{r=0}^{n} \binom{n}{r} F(3, 1#r)
+ \ldots + \sum_{r=0}^{n} \binom{n}{r} F(k, 1#r) + \ldots + \sum_{r=0}^{n} \binom{n}{r} F(\alpha, 1#r)

= \sum_{k=0}^{\alpha} \sum_{r=0}^{n} \binom{n}{r} F(k, 1#r)

This is a reduction formula for $F(\alpha, 1#(n+1))$

A Result of significance

From theorem (3.1) of Ref.: [2], we have

$$F'(p^a@1#(n+1)) = F(\alpha, 1#(n+1)) = \sum_{m=0}^{n} a_{(n+1,m)} F^{m*}(N)$$

where

$$a_{(n+1,m)} = \frac{1}{m!} \sum_{k=1}^{m} (-1)^{m-k} \cdot m^k \cdot \binom{m}{k} \cdot k^{n+1}$$

and

$$F^{m*}(p^a) = \sum_{k=0}^{\alpha} \binom{m+k-1}{m-1} P(\alpha-k)$$

This is the first result of some substance, giving a formula for
evaluating the number of Smarandache Factor Partitions of
numbers representable in a (one of the most simple) particular
canonical form. The complexity is evident. The challenging task
ahead for the readers is to derive similar expressions for other
canonical forms.
REFERENCE


[3] " The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texax at Austin, USA.