A LINEAR COMBINATION WITH SMARANDACHE FUNCTION TO OBTAIN THE IDENTITY

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In this paper we consider a numerical function \( i_p : \mathbb{N}^* \to \mathbb{N} \) (\( p \) is an arbitrary prime number) associated with a particular Smarandache Function \( S_p : \mathbb{N}^* \to \mathbb{N} \) such that \( (1/p)S_p(a) + i_p(a) = a \).

1. INTRODUCTION. In [7] is defined a numerical function \( S : \mathbb{N}^* \to \mathbb{N} \), \( S(n) \) is the smallest integer such that \( S(n)! \) is divisible by \( n \). This function may be extended to all integers by defining \( S(-n) = S(n) \).

If \( a \) and \( b \) are relatively prime then \( S(a \cdot b) = \max\{S(a), S(b)\} \), and if \( [a, b] \) is the last common multiple of \( a \) and \( b \) then \( S([a, b]) = \max\{S(a), S(b)\} \).

Suppose that \( n = p_1^e p_2^e \cdots p_r^e \) is the factorization of \( n \) into primes. In this case,

\[
S(n) = \max\{S(p_1^e), \ldots, S(p_r^e)\} \tag{1}
\]

Let \( a_n(p) = (p^n - 1) / (p - 1) \) and \([p]\) be the generalized numerical scale generated by \( (a_n(p))_{n \in \mathbb{N}} : \)

\[
[p] : a_1(p), a_2(p), \ldots, a_n(p), \ldots
\]

By \([p]\) we shall note the standard scale induced by the net \( b_n(p) = p^n : \)

\[
(p) : 1, p, p^2, p^3, \ldots, p^n, \ldots
\]

In [2] it is proved that

\[
S(p^e) = p\left(\left\lfloor \log_{p^e} \left(\frac{p-1}{p-1}\cdot a + 1\right) \right\rfloor\right) \tag{2}
\]

That is the value of \( S(p^e) \) is obtained multiplying by \( p \) the number obtained writing the exponent \( a \) in the generalized scale \([p]\) and "reading" it in the standard scale \((p)\).

Let us observe that the calculus in the generalized scale \([p]\) is different from the calculus in the standard scale \((p)\), because

\[
a_{n+1}(p) = pa_n(p) + 1 \quad \text{and} \quad b_{n+1}(p) = pb_n(p) \tag{3}
\]

We have also

\[
a_m(p) \leq a \Leftrightarrow (p^m - 1) / (p - 1) \leq a \Leftrightarrow p^m \leq (p - 1) \cdot a + 1 \Leftrightarrow m \leq \log_p ((p - 1) \cdot a + 1)
\]

so if

\[
a_{[p]} = v_1a_1(p) + v_{i-1}a_{i-1}(p) + \ldots + v_1a_1(p) = v_1v_1v_1\ldots v_{[p]}
\]

is the expression of \( a \) in the scale \([p]\) then \( t \) is the integer part of \( \log_p ((p - 1) \cdot a + 1) \)

\[
t = \left\lfloor \log_p ((p - 1) \cdot a + 1) \right\rfloor
\]

and the digit \( v_t \) is obtained from \( a = v_t a_t(p) + r_{i-t} \).

In [1] it is proved that

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\[ S(p') = (p-1) \cdot a + \sigma_{[p]}(a) \]  
(4)

where \( \sigma_{[p]}(a) = v_1 + v_2 + \ldots + v_v \).

A Legendre formula asserts that

\[ a! = \prod_{p | a} E_p(a) \]

where \( E_p(a) = \sum_{p | a} \left\lfloor \frac{a}{p^j} \right\rfloor \).

We have also that (5)

\[ E_p(a) = \frac{a - \sigma_{[p]}(a)}{p-1} \]

and (11) \( E_p(a) = \left( \left\lfloor \frac{a}{p} \right\rfloor \right) \).

In [1] is given also the following relation between the function \( E_p \) and the Smarandache function

\[ S(p') = \frac{(p-1)^2}{p} (E_p(a) + a + \sigma_{[p]}(a)) + \sigma_{[p]}(a) + \sigma_{[p]}(a) \]

There exist a great number of problems concerning the Smarandache function. We present some of these problems.

P. Gronas find (3) the solution of the diophantine equation \( F_s(n) = n \), where \( F_s(n) = \sum_{d \mid n} S(d) \). The solution are \( n = 9, n = 16 \) or \( n = 24, \) or \( n = 2p \), where \( p \) is a prime number.

T. Yau (8) find the triplets which verifies the Fibonacci relationship

\[ S(n) = S(n + 1) + S(n + 2) \]

Checking the first 1200 numbers, he find just two triplets which verifies this relationship: (9, 10, 11) and (119, 120, 121). He can’t find theoretical proof.

The following conjecture that: “the equation \( S(x) = S(x + 1) \), has no solution”, was not completely solved until now.

2. The Function \( i_p(a) \). In this section we shall note \( S(p') = S_p(a) \). From the Legendre formula it results (4) that

\[ S_p(a) = p(a - i_p(a)) \quad \text{with} \quad 0 \leq i_p(a) \leq \left\lfloor \frac{a-1}{p} \right\rfloor . \]

(6)

That is we have

\[ \frac{1}{p} S_p(a) + i_p(a) = a \]

(7)

and so for each function \( S_p \) there exists a function \( i_p \) such that we have the linear combination (7) to obtain the identity.

In the following we keep out some formulae for the calculus of \( i_p \). We shall obtain a duality relation between \( i_p \) and \( E_p \).

Let \( a_p = u_1 u_2 \ldots u_k u_0 = u_1 p^k + u_2 p^{k-1} + \ldots + u_k p + u_0 \). Then

\[ 79 \]
\[ a = (p-1) \left( \frac{\frac{p^k-1}{p-1} + \frac{p^{k-1}-1}{p-1} + \ldots + \frac{p-1}{p-1}}{} + (u_k + u_{k-1} + \ldots + u_1) + u_0 = 
\right) 
\]

\[ (p-l) \left( \begin{bmatrix} a \\ p \end{bmatrix} \right) + \sigma_{p} (a) = (p-1)E_p (a) + \sigma_{p} (a) \]  

(8)

From (4) it results

\[ a = \frac{S_p (a) - \sigma_{p} (a)}{p-1} \]  

(9)

From (8) and (9) we deduce

\[ (p-l)E_p (a) + \sigma_{p} (a) = \frac{S_p (a) - \sigma_{p} (a)}{p-1} \]  

So,

\[ S_p (a) = (p-1)^2 E_p (a) + (p-1)\sigma_{p} (a) + \sigma_{p} (a) \]  

(10)

From (4) and (7) it results

\[ i_p (a) = \frac{a - \sigma_{p} (a)}{p} \]  

(11)

and it is easy to observe a complementary with the equality (5).

Combining (5) and (11) it results

\[ i_p (a) = \frac{(p-l)E_p (a) + \sigma_{p} (a) - \sigma_{p} (a)}{p} \]  

(12)

From

\[ a = v_1, v_{r-1}, \ldots, v_{[p]} = v_1 (p^{r-1} + p^{r-2} + \ldots + p + 1) + v_{r-1} (p^{r-2} + p^{r-3} + \ldots + p + 1) + \ldots + v_2 (p + 1) + v_1 \]  

it results that

\[ a = (v_1 p^{r-1} + v_{r-1} p^{r-2} + \ldots + v_2 p + v_1) + v_1 (p^{r-2} + p^{r-3} + \ldots + p + 1) + v_{r-1} (p^{r-3} + p^{r-4} + \ldots + 1) + \ldots + v_2 (p + 1) + v_1 \]

\[ v_1 (p + 1) + v_2 = \left( a_{[p]} \right)_{(p)} + \left[ \begin{bmatrix} a \\ p \end{bmatrix} - \frac{\sigma_{p} (a)}{p} \right] \]  

because

\[ \left[ \begin{bmatrix} a \\ p \end{bmatrix} = \left[ v_1 (p^{r-2} + p^{r-3} + \ldots + p + 1) + \frac{v_1}{p} + v_{r-1} (p^{r-3} + p^{r-4} + \ldots + p + 1) + \frac{v_{r-1}}{p} + \ldots + \right. \right. \]

\[ + v_2 (p + 1) + \frac{v_2}{p} + v_1 + \frac{v_1}{p} \right] = v_1 (p^{r-2} + p^{r-3} + \ldots + p + 1) + \]

\[ + v_{r-1} (p^{r-3} + p^{r-4} + \ldots + p + 1) + \ldots + v_2 (p + 1) + v_1 + \left[ \frac{\sigma_{p} (a)}{p} \right] \]

we have \([n+x] = n + [x] \).

Then

\[ a = \left( a_{[p]} \right)_{(p)} + \left[ \begin{bmatrix} a \\ p \end{bmatrix} - \frac{\sigma_{p} (a)}{p} \right] \]  

(13)

or

\[ a = \frac{S_p (a)}{p} + \left[ \begin{bmatrix} a \\ p \end{bmatrix} - \frac{\sigma_{p} (a)}{p} \right] \]

It results that

\[ 80 \]
From (11) and (14) we obtain

\[ i_p(a) = \left( a - \left[ \frac{\sigma_{|p|}(a)}{p} \right] \right) \]  

(14)

It is known that there exists \( m, n \in \mathbb{N} \) such that the relation

\[ \left[ \frac{m-n}{p} \right] = \left[ \frac{m}{p} \right] - \left[ \frac{n}{p} \right] \]

is not verified.

But if \( \frac{m-n}{p} \in \mathbb{N} \) then the relation (16) is satisfied.

From (11) and (15) it results

\[ \frac{a - \sigma_{|p|}(a)}{p} = \left( a \right) - \left[ \frac{\sigma_{|p|}(a)}{p} \right] \]

This equality results also by the fact that \( i_p(a) \in \mathbb{N} \).

From (2) and (11) or from (13) and (15) it results that

\[ i_p(a) = a - \left( a_{|p|} \right) \]  

(17)

From the condition on \( i_p \) in (6) it results that \( \Delta = \left[ \frac{a-1}{p} \right] - i_p(a) \geq 0 \).

To calculate the difference \( \Delta = \left[ \frac{a-1}{p} \right] - i_p(a) \) we observe that

\[ \Delta = \left[ \frac{a-1}{p} \right] - i_p(a) = \left[ \frac{a-1}{p} \right] - \left[ \frac{a}{p} \right] + \left[ \frac{\sigma_{|p|}(a)}{p} \right] \]  

(18)

For \( a \in [kp+1, kp+p-1] \) we have \( \left[ \frac{a-1}{p} \right] = \left[ \frac{a}{p} \right] \) so

\[ \Delta = \left[ \frac{a-1}{p} \right] - i_p(a) = \left[ \frac{\sigma_{|p|}(a)}{p} \right] \]  

(19)

If \( a = kp \) then \( \left[ \frac{a-1}{p} \right] = \left[ \frac{kp-1}{p} \right] = \left[ \frac{k-1}{p} \right] = k-1 \) and \( \left[ \frac{a}{p} \right] = k \).

So, (18) becomes

\[ \Delta = \left[ \frac{a-1}{p} \right] - i_p(a) = \left[ \frac{\sigma_{|p|}(a)}{p} \right] - 1 \]  

(20)

Analogously, if \( a = kp + p \), we have

\[ \left[ \frac{a-1}{p} \right] = \left[ \frac{p(k+1)-1}{p} \right] = \left[ \frac{k+1-1}{p} \right] = k \) and \( \left[ \frac{a}{p} \right] = k+1 \)

so, (18) has the form (20).

For any number \( a \), for which \( \Delta \) is given by (19) or by (20), we deduce that \( \Delta \) is maximum when \( \sigma_{|p|}(a) \) is maximum, so when

\[ a_M = \prod_{t=0}^{n} \left( p - 1 \right) \]  

(21)
That is
\[ a_M = (p-1)a_t(p) + (p-1)a_{t-1}(p) + \ldots + (p-1)a_2(p) + p = \]
\[ = (p-1)\left(\frac{p^t-1}{p-1} + \frac{p^{t-1}-1}{p-1} + \ldots + \frac{p^2-1}{p-1}\right) + p = \]
\[ = (p^t + p^{t-1} + \ldots + p^2 + p) - (t-1) = pa_t(p) - (t-1) \]

It results that \(a_M\) is not multiple of \(p\) if and only if \(t-1\) is not a multiple of \(p\).

In this case \(\sigma_{[p]}(a) = (t-1)(p-1) + p = pt - t + 1\) and
\[ \Delta = \left[\frac{\sigma_{[p]}(a)}{p}\right] = \left[t - \frac{t-1}{p}\right] = t - \left[\frac{t-1}{p}\right]. \]

So \(i_p(a_M) \geq \left[\frac{a_m-1}{p}\right] - t\) or \(i_p(a_M) \in \left[\frac{a_m-1}{p}\right] - t, \left[\frac{a_m-1}{p}\right] \right].\) If \(t-1 \in (kp, kp+p)\) then
\[ \left[\frac{t-1}{p}\right] = k \text{ and } k(p-1)+1 < \Delta(a_M) < k(p-1)+p+1 \text{ so } \lim_{a_M \to \infty} \Delta(a_M) = \infty. \]

We also observe that
\[ \left[\frac{a_m-1}{p}\right] = a_t(p) - \left[\frac{t-1}{p}\right] = \left[\frac{p^{t+1}-1}{p-1}\right] - \left[\frac{t-1}{p}\right] \in \left[\frac{p^{kp+1}-1}{p-1}\right] - k, \left[\frac{p^{kp+p+1}-1}{p-1}\right] - k. \]

Then if \(a_M \to \infty\) (as \(p^t\)), it results that \(\Delta(a_m) \to \infty\) (as \(x\)).

From \(i_p(a_m) = \frac{a_t(p) - \left[\frac{t-2}{p}\right]}{a_t(p) - \left[\frac{t-1}{p}\right]} \to 1\) it results \(\lim_{a \to [a-1]} i_p(a) = 1.\)

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BIBLIOGRAPHY


