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The Line n-Sigraph of a Symmetric n-Sigraph-IV

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Abstract: An *n-tuple* $(a_1, a_2, \cdots, a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, \cdots, a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. A*symmetric n-sigraph (symmetric n-marked graph)* is an ordered pair $S_n = (G, \sigma)$ ($S_n = (G, \mu)$), where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \to H_n$ ($\mu : V \to H_n$) is a function. In Bagga et al. (1995) introduced the concept of the *super line graph of index* r of a graph G, denoted by $\mathcal{L}_r(G)$. The vertices of $\mathcal{L}_r(G)$ are the r-subsets of $E(G)$ and two vertices P and Q are adjacent if there exist $p \in P$ and $q \in Q$ such that p and q are adjacent edges in G. Analogously, one can define the *super line symmetric n-sigraph of index* r of a symmetric n-sigraph $S_n = (G, \sigma)$ as a symmetric n-sigraph $\mathcal{L}_r(S_n) = (\mathcal{L}_r(G), \sigma')$, where $\mathcal{L}_r(G)$ is the underlying graph of $\mathcal{L}_r(S_n)$, where for any edge PQ in $\mathcal{L}_r(S_n)$, $\sigma'(PQ) = \sigma(P)\sigma(Q)$. It is shown that for any symmetric n-sigraph S_n, its $\mathcal{L}_r(S_n)$ is i-balanced and we offer a structural characterization of super line symmetric n-sigraphs of index r. Further, we characterize symmetric n-sigraphs S_n for which $S_n \sim \mathcal{L}_2(S_n)$, $\mathcal{L}_2(S_n) \sim L(S_n)$ and $\mathcal{L}_2(S_n) \sim \overline{S_n}$ where $\sim$ denotes switching equivalence and $\mathcal{L}_2(S_n)$, $L(S_n)$ and $\overline{S_n}$ are denotes the super line symmetric n-sigraph of index 2, line symmetric n-sigraph and complementary symmetric n-sigraph of S_n respectively. Also, we characterize symmetric n-sigraphs S_n for which $S_n \cong \mathcal{L}_2(S_n)$ and $\mathcal{L}_2(S_n) \cong L(S_n)$.

Key Words: Smarandachely symmetric n-marked graph, symmetric n-sigraph, symmetric n-marked graph, balance, switching, balance, super line symmetric n-sigraph, line symmetric n-sigraph.

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§1. Introduction

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [6]. We consider only finite, simple graphs free from self-loops.

Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, \cdots, a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, \cdots, a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the

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order of H_n is 2^m, where $m = \lceil \frac{n}{2} \rceil$.

A *Smarandachely k-marked graph* is an ordered pair $S = (G, \mu)$ where $G = (V, E)$ is a graph called *underlying graph of S* and $\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ is a function, where each $\overline{e}_i \in \{+, -\}$. An *$n$-tuple* $(a_1, a_2, ..., a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. A *Smarandachely symmetric n-marked graph* is an ordered pair $S_n = (G, \mu)$, where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\mu : V \to H_n$ is a function. Particularly, a Smarandachely 2-marked graph is called a *symmetric n-sigraph (symmetric n-marked graph)*, where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \to H_n$ ($\mu : V \to H_n$) is a function.

In this paper by an *n-tuple/n-sigraph/n-marked graph* we always mean a symmetric n-tuple/symmetric n-sigraph/symmetric n-marked graph.

An n-tuple $(a_1, a_2, \cdots, a_n)$ is the *identity n-tuple*, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple*. In an n-sigraph $S_n = (G, \sigma)$ an edge labelled with the identity n-tuple is called an *identity edge*, otherwise it is a *non-identity edge*.

Further, in an n-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the *n-tuple* $\sigma(A)$ is the product of the n-tuples on the edges of A.

In [12], the authors defined two notions of balance in n-sigraph $S_n = (G, \sigma)$ as follows (See also R. Rangarajan and P.S.K.Reddy [9]:

Definition 1.1 *Let $S_n = (G, \sigma)$ be an n-sigraph. Then,*

(i) *S_n is* identity balanced *(or* i-balanced*), if product of n-tuples on each cycle of S_n is the identity n-tuple, and*

(ii) *S_n is* balanced*, if every cycle in S_n contains an even number of non-identity edges.*

Note An i-balanced n-sigraph need not be balanced and conversely.

The following characterization of i-balanced n-sigraphs is obtained in [12].

Proposition 1.1(E. Sampathkumar et al. [12]) *An n-sigraph $S_n = (G, \sigma)$ is i-balanced if, and only if, it is possible to assign n-tuples to its vertices such that the n-tuple of each edge uv is equal to the product of the n-tuples of u and v.*

In [12], the authors also have defined switching and cycle isomorphism of an n-sigraph $S_n = (G, \sigma)$ as follows (See also [7,10,11] & [14]-[18]):

Let $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$, be two n-sigraphs. Then S_n and S'_n are said to be *isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that if uv is an edge in S_n with label $(a_1, a_2, \cdots, a_n)$ then $\phi(u)\phi(v)$ is an edge in S'_n with label $(a_1, a_2, \cdots, a_n)$.

Given an n-marking μ of an n-sigraph $S_n = (G, \sigma)$, *switching* S_n with respect to μ is the operation of changing the n-tuple of every edge uv of S_n by $\mu(u)\sigma(uv)\mu(v)$. The n-sigraph obtained in this way is denoted by $\mathcal{S}_\mu(S_n)$ and is called the *μ-switched n-sigraph* or just *switched n-sigraph*.

Further, an n-sigraph S_n *switches* to n-sigraph S'_n (or that they are *switching equivalent* to each other), written as $S_n \sim S'_n$, whenever there exists an n-marking of S_n such that $\mathcal{S}_\mu(S_n) \cong S'_n$.

Two n-sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that the n-tuple $\sigma(C)$ of every cycle C in S_n equals to the n-tuple $\sigma(\phi(C))$ in S'_n. We make use of the following known result (see [12]).

Proposition 1.2(E. Sampathkumar et al. [12]) *Given a graph G, any two n-sigraphs with G as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

In this paper, we introduced the notion called super line n-sigraph of index r and we obtained some interesting results in the following sections. The super line n-sigraph of index r is the generalization of line n-sigraph.

§2. Super Line n-Sigraph $\mathcal{L}_r(S_n)$

In [1], the authors introduced the concept of the *super line graph*, which generalizes the notion of line graph. For a given G, its super line graph $\mathcal{L}_r(G)$ of index r is the graph whose vertices are the r-subsets of $E(G)$, and two vertices P and Q are adjacent if there exist $p \in P$ and $q \in Q$ such that p and q are adjacent edges in G. In [1], several properties of $\mathcal{L}_r(G)$ were studied. Many other properties and concepts related to super line graphs were presented in [2,4]. The study of super line graphs continues the tradition of investigating generalizations of line graphs in particular and of graph operators in general, as elaborated in the classical monograph by Prisner [8]. From the definition, it turns out that $\mathcal{L}_1(G)$ coincides with the line graph $L(G)$. More specifically, some results regarding the super line graph of index 2 were presented in [3] and [5]. Several variations of the super line graph have been considered.

In this paper, we extend the notion of $\mathcal{L}_r(G)$ to realm of n-sigraphs as follows: The *super line n-sigraph of index r* of an n-sigraph $S_n = (G, \sigma)$ as an n-sigraph $\mathcal{L}_r(S_n) = (\mathcal{L}_r(G), \sigma')$, where $\mathcal{L}_r(G)$ is the underlying graph of $\mathcal{L}_r(S_n)$, where for any edge PQ in $\mathcal{L}_r(S_n)$, $\sigma'(PQ) = \sigma(P)\sigma(Q)$.

Hence, we shall call a given n-sigraph S_n a *super line n-sigraph of index r* if it is isomorphic to the super line n-sigraph of index r, $\mathcal{L}_r(S'_n)$ of some n-sigraph S'_n. In the following subsection, we shall present a characterization of super line n-sigraph of index r.

The following result indicates the limitations of the notion $\mathcal{L}_r(S_n)$ as introduced above, since the entire class of i-unbalanced n-sigraphs is forbidden to be super line n-sigraphs of index r.

Proposition 2.1 *For any n-sigraph $S_n = (G, \sigma)$, its $\mathcal{L}_r(S_n)$ is i-balanced.*

Proof Let σ' denote the n-tuple of $\mathcal{L}_r(S_n)$ and let the n-tuple σ of S_n be treated as an n-marking of the vertices of $\mathcal{L}_r(S_n)$. Then by definition of $\mathcal{L}_r(S_n)$ we see that $\sigma'(PQ) = \sigma(P)\sigma(Q)$, for every edge PQ of $\mathcal{L}_r(S_n)$ and hence, by Proposition 1.1, the result follows. □

Corollary 2.2 *For any n-sigraph $S_n = (G, \sigma)$, its $\mathcal{L}_2(S_n)$ is i-balanced.*

For any positive integer k, the k^{th} iterated super line n-sigraph of index r, $\mathcal{L}_r(S_n)$ of S_n is defined as follows:

$$\mathcal{L}_r^0(S_n) = S_n,\ \mathcal{L}_r^k(S_n) = \mathcal{L}_r(\mathcal{L}_r^{k-1}(S_n))$$

Corollary 2.3 *For any n-sigraph $S_n = (G, \sigma)$ and any positive integer k, $\mathcal{L}_r^k(S_n)$ is i-balanced.*

The *line graph* $L(G)$ of graph G has the edges of G as the vertices and two vertices of $L(G)$ are adjacent if the corresponding edges of G are adjacent. The *line n-sigraph* of an n-sigraph $S_n = (G, \sigma)$ is an n-sigraph $L(S_n) = (L(G), \sigma')$, where for any edge ee' in $L(S_n)$, $\sigma'(ee') = \sigma(e)\sigma(e')$. This concept was introduced by E. Sampatkumar et al. [13]. The following result is one can easily deduce from Proposition 2.1.

Corollary 2.4 (E. Sampathkumar et al. [13]) *For any n-sigraph $S_n = (G, \sigma)$, its line n-sigraph $L(S_n)$ is i-balanced.*

In [5], the authors characterized those graphs that are isomorphic to their corresponding super line graphs of index 2.

Proposition 2.5(K. S. Bagga et al. [5]) *For a graph $G = (V, E)$, $G \cong \mathcal{L}_2(G)$ if, and only if, $G = K_3$.*

We now characterize the n-sigraphs that are switching equivalent to their super line n-sigraphs of index 2.

Proposition 2.6 *For any n-sigraph $S_n = (G, \sigma)$, $S_n \sim \mathcal{L}_2(S_n)$ if, and only if, $G = K_3$ and S is i-balanced n-sigraph.*

Proof Suppose $S_n \sim \mathcal{L}_2(S_n)$. This implies, $G \cong \mathcal{L}_2(G)$ and hence G is K_3. Now, if S_n is any n-sigraph with underlying graph as K_3, Corollary 2.2 implies that $\mathcal{L}_2(S_n)$ is i-balanced and hence if S_n is i-unbalanced and its $\mathcal{L}_2(S_n)$ being i-balanced can not be switching equivalent to S_n in accordance with Proposition 1.2. Therefore, S_n must be i-balanced.

Conversely, suppose that S_n is i-balanced n-sigraph and G is K_3. Then, since $\mathcal{L}_2(S_n)$ is i-balanced as per Corollary 2.2 and since $G \cong \mathcal{L}_2(G)$, the result follows from Proposition 1.2 again. □

We now characterize the n-sigraphs that are isomorphic to their super line n-sigraphs of index 2.

Proposition 2.7 *For any n-sigraph $S_n = (G, \sigma)$, $S_n \cong \mathcal{L}_2(S_n)$ if, and only if, $G = K_3$ and S_n is i-balanced n-sigraph.*

In [5], the authors characterized whose super line graphs of index 2 that are isomorphic to $L(G)$.

Proposition 2.8(K. S. Bagga et al. [5]) *For a graph $G = (V, E)$, $\mathcal{L}_2(G) \cong L(G)$ if, and only if, G is $K_{1,3}$, K_3 or $3K_2$.*

From the above result we have following result for signed graphs:

Proposition 2.9 *For any n-sigraph $S_n = (G, \sigma)$, $\mathcal{L}_2(S_n) \sim L(S_n)$ if, and only if, G is $K_{1,3}$, K_3 or $3K_2$.*

Proof Suppose $\mathcal{L}_2(S_n) \sim L(S_n)$. This implies, $\mathcal{L}_2(G) \cong L(G)$ and hence by Proposition 2.8, we see that the graph G must be isomorphic to $K_{1,3}$, K_3 or $3K_2$.

Conversely, suppose that G is a $K_{1,3}$, K_3 or $3K_2$. Then $\mathcal{L}_2(G) \cong L(G)$ by Proposition 2.8. Now, if S_n any n-sigraph on any of these graphs, By Proposition 2.1 and Corollary 2.4, $\mathcal{L}_2(S_n)$ and $L(S_n)$ are i-balanced and hence, the result follows from Proposition 1.2. □

We now characterize n-sigraphs whose super line n-sigraphs $\mathcal{L}_2(S_n)$ that are isomorphic to line n-sigraphs.

Proposition 2.10 *For any n-sigraph $S_n = (G, \sigma)$, $\mathcal{L}_2(S_n) \cong L(S_n)$ if, and only if, G is $K_{1,3}$, K_3 or $3K_2$.*

Proof Clearly $\mathcal{L}_2(G) \cong L(G)$, when G is $K_{1,3}$, K_3 or $3K_2$. Consider the map $f : V(\mathcal{L}_2(G)) \to V(L(G))$ defined by $f(e_1e_2, e_2e_3) = (e_1, e_3)$ is an isomorphism. Let σ be any n-tuple on $K_{1,3}$, K_3 or $3K_2$. Let $e = (e_1e_2, e_2e_3)$ be an edge in $\mathcal{L}_2(G)$, where G is $K_{1,3}$, K_3 or $3K_2$. Then the n-tuple of the edge e in $\mathcal{L}_2(G)$ is the $\sigma(e_1e_2)\sigma(e_2e_3)$ which is the n-tuple of the edge (e_1, e_3) in $L(G)$, where G is $K_{1,3}$, K_3 or $3K_2$. Hence the map f is also an n-sigraph isomorphism between $\mathcal{L}_2(S_n)$ and $L(S_n)$. □

Let $S_n = (G, \sigma)$ be an n-sigraph. *The complement* of S_n is an n-sigraph $\overline{S_n} = (\overline{G}, \sigma^c)$, where $\overline{G}$ is the underlying graph of $\overline{S_n}$ and for any edge $e = uv \in \overline{S_n}$, $\sigma^c(uv) = \mu(u)\mu(v)$, where for any $v \in V$, $\mu(v) = \prod\limits_{u \in N(v)} \sigma(uv)$. Clearly, $\overline{S_n}$ as defined here is an i-balanced n-sigraph due to Proposition 1.1.

In [5], the authors proved there are no solutions to the equation $\mathcal{L}_2(G) \sim \overline{G}$. So it is impossible to construct switching equivalence relation of $\mathcal{L}_2(S_n) \sim \overline{S_n}$ for any arbitrary n-sigraph. The following result characterizes n-sigraphs which are super line n-sigraphs of index r.

Proposition 2.11 *An n-sigraph $S_n = (G, \sigma)$ is a super line n-sigraph of index r if and only if S_n is i-balanced n-sigraph and its underlying graph G is a super line graph of index r.*

Proof Suppose that S_n is i-balanced and G is a $\mathcal{L}_r(G)$. Then there exists a graph H such that $\mathcal{L}_r(H) \cong G$. Since S_n is i-balanced, by Proposition 1.1, there exists an n-marking μ of G such that each edge uv in S_n satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the n-sigraph $S'_n = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the n-marking of the corresponding vertex in G. Then clearly, $\mathcal{L}_r(S'_n) \cong S_n$. Hence S_n is a super line n-sigraph of index r.

Conversely, suppose that $S_n = (G, \sigma)$ is a super line n-sigraph of index r. Then there exists an n-sigraph $S'_n = (H, \sigma')$ such that $\mathcal{L}_r(S'_n) \cong S_n$. Hence G is the $\mathcal{L}_r(G)$ of H and by Proposition 2.1, S_n is i-balanced. □

If we take $r = 1$ in $\mathcal{L}_r(S_n)$, then this is the ordinary line n-sigraph. In [13], the authors obtained structural characterization of line n-sigraphs and clearly Proposition 2.11 is the generalization of line n-sigraphs.

Proposition 2.12(E. Sampathkumar et al. [13]) *An n-sigraph $S_n = (G, \sigma)$ is a line n-sigraph if, and only if, S_n is i-balanced n-sigraph and its underlying graph G is a line graph.*

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