**On the mean value of the Smarandache ceil function**

Wang Xiaoying
Research Center for Basic Science, Xi’an Jiaotong University
Xi’an, Shaanxi, P.R.China

**Abstract** For any fixed positive integer $n$, the Smarandache ceil function of order $k$ is denoted by $N^* \rightarrow N$ and has the following definition:

$$S_k(n) = \min\{x \in \mathbb{N} : n \mid x^k\}, \quad \forall n \in N^*.$$

In this paper, we study the mean value properties of the Smarandache ceil function, and give a sharp asymptotic formula for it.

**Keywords** Smarandache ceil function; Mean value; Asymptotic formula.

§1. Introduction

For any fixed positive integer $n$, the Smarandache ceil function of order $k$ is denoted by $N^* \rightarrow N$ and has the following definition:

$$S_k(n) = \min\{x \in \mathbb{N} : n \mid x^k\}, \quad \forall n \in N^*.$$

For example, $S_2(1) = 1$, $S_2(2) = 2$, $S_2(3) = 3$, $S_2(4) = 2$, $S_2(5) = 5$, $S_2(6) = 6$, $S_2(7) = 7$, $S_2(8) = 4$, $S_2(9) = 3$, \ldots. This was introduced by Smarandache who proposed many problems in [1]. There are many papers on the Smarandache ceil function. For example, Ibstedt [2] [3] studied this function both theoretically and computationally, and got the following conclusions:

$$(a, b) = 1 \Rightarrow S_k(ab) = S_k(a)S_k(b), \quad a, b \in N^*.$$

$$S_k(p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}) = S_k(p_1^{a_1})\cdots S_k(p_r^{a_r}).$$

In this paper, we study the mean value properties of the Smarandache ceil function, and give a sharp asymptotic formula for it. That is, we shall prove the following:

**Theorem.** For any real number $x \geq 2$, we have the asymptotic formula

$$\sum_{n \leq x} \frac{1}{S_k(n)} = \frac{3}{2\pi^2} \ln^2 x + A_1 \ln x + A_2 + O(x^{-\frac{1}{2}} + \epsilon),$$

where $A_1$ and $A_2$ are two computable constants, $\epsilon$ is any fixed positive integer.

---

1This work is supported by the N.S.F(10271093) and P.N.S.F of P.R.China.
§2. Proof of the theorem

To complete the proof of the theorem, we need the following Lemma, which is called the Perron’s formula (See reference [4]):

**Lemma.** Suppose that the Dirichlet series \( f(s) = \sum_{n=1}^{\infty} a(n)n^{-s}, \) convergent absolutely for \( \sigma > \sigma_a, \) and that there exist a positive increasing function \( H(u) \) and a function \( B(u) \) such that

\[
a(n) \leq H(n), \quad n = 1, 2, \cdots,
\]

and

\[
\sum_{n=1}^{\infty} |a(n)| n^{-\sigma} \leq B(\sigma), \quad \sigma > \sigma_a.
\]

Then for any \( s_0 = \sigma_0 + it_0, b_0 > \sigma_a, T \geq 1 \) and \( x \geq 1, x \) not to be an integer, we have

\[
\sum_{n \leq x} a(n)n^{-s_0} = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s_0 + s) \frac{x^s}{s} ds + O \left( \frac{x^b B(b + \sigma_0)}{T} \right) + O \left( x^{1-\sigma_0} H(2x) \min \left( 1, \frac{\log x}{T} \right) \right) + O \left( x^{-\sigma_0} H(N) \min \left( 1, \frac{x}{T \| x \|} \right) \right),
\]

where \( N \) is the nearest integer to \( x, \| x \| = |N - x|. \)

Now we complete the proof of the theorem. Let \( s = \sigma + it \) be a complex number and

\[
f(s) = \sum_{n=1}^{\infty} \frac{S_2(n)}{n^s},
\]

Note that \( \left| \frac{1}{S_2(n)} \right| \leq \frac{1}{\sqrt{n}}, \) so it is clear that \( f(s) \) is a Dirichlet series absolutely convergent for \( \text{Re}(s) > \frac{1}{2}, \) by Euler product formula [5] and the definition of \( S_2(n) \) we have

\[
f(s) = \prod_p \left( 1 + \frac{1}{S_2(p)p^s} + \frac{1}{S_2(p^2)p^{2s}} + \frac{1}{S_2(p^3)p^{3s}} + \frac{1}{S_2(p^4)p^{4s}} + \cdots \right)
\]

\[
= \prod_p \left( 1 + \frac{1}{p^s + 1} + \frac{1}{p^{2s} + 1} + \frac{1}{p^{3s} + 1} + \frac{1}{p^{4s} + 1} + \cdots \right)
\]

\[
= \prod_p \frac{1}{1 - \frac{1}{p^{s+1}}} \left( 1 + \frac{1}{p^{s+1}} \right)
\]

\[
= \frac{\zeta(2s+1)\zeta(s+1)}{\zeta(2s+2)},
\]

where \( \zeta(s) \) is the Riemann zeta-function and \( \prod_p \) denotes the product over all primes.
Taking

\[ H(x) = 1; \quad B(\sigma) = \frac{2}{2\sigma - 1}; \quad \sigma > \frac{1}{2}; \]

\[ s_0 = 0; \quad b = 1; \quad T = x^{\frac{5}{4}} \] in the above Lemma we may get

\[
\sum_{n \leq x} \frac{1}{S_2(n)} = \frac{1}{2\pi i} \int_{1+-ix^{\frac{5}{4}}}^{1+i+ix^{\frac{5}{4}}} f(s) \frac{x^s}{s} ds + O(x^{-\frac{1}{4}+\epsilon}).
\]

To estimate the main term, we move the integral line in the above formula from \( s = 1 \pm ix^{\frac{5}{4}} \) to \( s = -\frac{1}{4} \pm ix^{\frac{5}{4}} \). This time, the function \( f(s) \frac{x^s}{s} \) have a third order pole point at \( s = 0 \) with residue

\[
\frac{3}{2\pi^2} \ln^2 x + A_1 \ln x + A_2,
\]

where \( A_1 \) and \( A_2 \) are two computable constants.

Hence, we have

\[
\frac{1}{2\pi i} \left( \int_{1+-ix^{\frac{5}{4}}}^{-\frac{1}{4}+ix^{\frac{5}{4}}} + \int_{1+ix^{\frac{5}{4}}}^{-\frac{1}{4}+ix^{\frac{5}{4}} \pm \frac{1}{4}-ix^{\frac{5}{4}}} + \int_{-\frac{1}{4}-ix^{\frac{5}{4}}}^{1+ix^{\frac{5}{4}}} \right) \frac{\zeta(2s + 1)\zeta(s + 1)x^s}{\zeta(2s + 2)s} ds = \frac{3}{2\pi^2} \ln^2 x + A_1 \ln x + A_2.
\]

We can easily get the estimate

\[
\left| \frac{1}{2\pi i} \left( \int_{1+ix^{\frac{5}{4}}}^{-\frac{1}{4}+ix^{\frac{5}{4}}} + \int_{1+ix^{\frac{5}{4}}}^{-\frac{1}{4}+ix^{\frac{5}{4}}} \pm \frac{1}{4}-ix^{\frac{5}{4}}} + \int_{-\frac{1}{4}-ix^{\frac{5}{4}}}^{1+ix^{\frac{5}{4}}} \right) \frac{\zeta(2s + 1)\zeta(s + 1)x^s}{\zeta(2s + 2)s} ds \right| \ll x^{-\frac{1}{4}+\epsilon}.
\]

From above we may immediately get the asymptotic formula:

\[
\sum_{n \leq x} \frac{1}{S_2(n)} = \frac{3}{2\pi^2} \ln^2 x + A_1 \ln x + A_2 + O(x^{-\frac{1}{4}+\epsilon}).
\]

This completes the proof of the theorem.

**References**


[2] Ibstedt, Surfing on the ocean of numbers—a few Smarandache notions and similar topics, Erhus University press, New Mexico, 1997.

