The Smarandache minimum and maximum functions

József Sándor

Babes-Bolyai University of Cluj, Romania

Abstract This papers deals with the introduction and preliminary study of the Smarandache minimum and maximum functions.

Keywords Smarandache minimum and maximum functions; arithmetical properties.

1. Let \( f : \mathbb{N}^* \rightarrow \mathbb{N} \) be a given arithmetic function and \( A \subset \mathbb{N} \) a given set. The arithmetic function

\[
F^A_f(n) = \min \{ k \in A : n \mid f(k) \} \tag{1}
\]

has been introduced in [4] and [5].

For \( A = \mathbb{N}, f(k) = k! \) one obtains the Smarandache function; For \( A = \mathbb{N}^*, A = p = \{2, 3, 5, \ldots\} = \) set of all primes, one obtains a function

\[
P(n) = \min \{ k \in P : n \mid k! \} \tag{2}
\]

For the properties of this function, see [4] and [5]. The “dual” function of (1) has been defined by

\[
G^A_g(n) = \max \{ k \in A : g(k) \mid n \}, \tag{3}
\]

where \( g : \mathbb{N}^* \rightarrow \mathbb{N} \) is a given function, and \( A \subset \mathbb{N} \) is a given set. Particularly, for \( A = \mathbb{N}^*, g(k) = k! \), one obtains the dual of the Smarandache function,

\[
S_*(n) = \max \{ k \geq 1 : k! \mid n \} \tag{4}
\]

For the properties of this function, see [4] and [5]. F.Luca [3], K.Atanassov [1] and L.le [2] have proved in the affirmative a conjecture of the author.

For \( A = \mathbb{N}^* \) and \( f(k) = g(k) = \varphi(k) \) in (1), resp.(3) one obtains the Euler minimum, resp. maximum-function, defined by

\[
E(n) = \min \{ k \geq 1 : n \mid \varphi(k) \}, \tag{5}
\]
For the properties of these function, see [6]. When $A = N^*$, $f(k) = d(k) =$number of divisors of $k$, one obtains the divisor minimum function (see [4], [5] and [7])

$$D(n) = \min\{k \geq 1 : n \mid d(k)\}. \tag{7}$$

It is interesting to note that the divisor maximum function (i.e., the “ dual” of $D(n)$) given by

$$D^*(n) = \max\{k \geq 1 : d(k) \mid n\} \tag{8}$$

is not well defined! Indeed, for any prime $p$ one has $d(p^{n-1}) = n \mid n$ and $p^{n-1}$ is unbounded as $p \to \infty$. For a finite set $A$, however $D^A(n)$ does exist. On one hand, it has been shown in [4] and [5] that

$$\sum(n) = \min\{k \geq 1 : n \mid \sigma(k)\} \tag{9}$$

is well defined. (Here $\sigma(k)$ denotes the sum of all divisors of $k$). The dual of the sum-of-divisors minimum function is

$$\sum^*(n) = \max\{k \geq 1 : \sigma(k) \mid n\} \tag{10}$$

Since $\sigma(1) = 1 \mid n$ and $\sigma(k) \geq k$, clearly $\sum^*(n) \leq n$, so this function is well defined (see [8]).

2. The Smarandache minimum function will be defined for $A = N^*$, $f(k) = S(k)$ in (1). Let us denote this function by $S_{\min}$:

$$S_{\min}(n) = \min\{k \geq 1 : n \mid S(k)\} \tag{11}$$

Let us assume that $S(1) = 1$, i. e., $S(n)$ is defined by (1) for $A = N^*$, $f(k) = k!$:

$$S(n) = \min\{k \geq 1 : n \mid k!\} \tag{12}$$

Otherwise (i.e.when $S(1) = 0$) by $n \mid 0$ for all $n$, by (11) for one gets the trivial function $S_{\min}(n) = 0$. By this assumption, however, one obtains a very interesting (and difficult) function $s_{\min}$ given by (11). Since $n \mid S(n!) = n$, this function is correctly defined.

The Smarandache maximum function will be defined as the dual of $S_{\min}$:

$$S_{\max}(n) = \max\{k \geq 1 : S(k) \mid n\}. \tag{13}$$

We prove that this is well defined. Indeed, for a fixed $n$, there are a finite number of divisors of $n$, let $i \mid n$ be one of them. The equation

$$S(k) = i \tag{14}$$

is well-known to have a number of $d(i!) - d((i - 1)!)$ solutions, i. e., in a finite number. This implies that for a given $n$ there are at most finitely many $k$ with $S(k) \mid k$, so the maximum in (13) is attained.
Clearly $S_{\text{min}}(1) = 1, S_{\text{min}}(2) = 2, S_{\text{min}}(3) = 3, S_{\text{min}}(4) = 4, S_{\text{min}}(5) = 5, S_{\text{min}}(6) = 9, S_{\text{min}}(7) = 7, S_{\text{min}}(8) = 32, S_{\text{min}}(9) = 27, S_{\text{min}}(10) = 25, S_{\text{min}}(11) = 11$, etc, which can be determined from a table of Smarandache numbers:

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>S(n)</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>4</td>
<td>13</td>
</tr>
</tbody>
</table>

We first prove that:

**Theorem 1.** $S_{\text{min}}(n) \geq n$ for all $n \geq 1$, with equality only for $n = 1, 4, p(p = \text{prime})$

**Proof.** Let $n | S(k)$. If we would have $k < n$, then since $S(k) \leq k < n$ we should get $S(k) < n$, in contradiction with $n | S(k)$. Thus $k \geq n$, and taking minimum, the inequality follows. There is equality for $n = 1$ and $n = 4$. Let now $n > 4$. If $n = p = \text{prime}$, then $p | S(p) = p$, but for $k < p$, $p \nmid S(k)$. Indeed, by $S(k) \leq k < p$ this is impossible. Reciprocally, if $\min\{k \geq 1 : n | S(k)\} = n$, then $n | S(n)$, and by $S(n) \leq n$ this is possible only when $S(n) = n$, i.e., when $n = 1, 4, p(p = \text{prime})$.

**Theorem 2.** For all $n \geq 1$,

$$S_{\text{min}}(n) \leq n! \leq S_{\text{max}}(n)$$

**Proof.** Since $S(n!)=n$, definition (11) gives the left side of (16), while definition (13) gives the right side inequality.

**Corollary.** The series $\sum_{n \geq 1} \frac{1}{S_{\text{min}}(n)}$ is divergent, while the series $\sum_{n \geq 1} \frac{1}{S_{\text{max}}(n)}$ is convergent.

**Proof.** Since $\sum_{n \geq 1} \frac{1}{S_{\text{max}}(n)} \leq \sum_{n \geq 1} \frac{1}{n!} = e - 1$ by (16), this series is convergent. On the other hand,

$$\sum_{n \geq 1} \frac{1}{S_{\text{min}}(n)} \geq \sum_{p} \frac{1}{S_{\text{min}}(p)} = \sum_{p} \frac{1}{p} = +\infty,$$

so the first series is divergent.

**Theorem 3.** For all primes $p$ one has

$$S_{\text{max}}(p) = p!$$

**Proof.** Let $S(k) | p$. Then $S(k) = 1$ or $S(k) = p$. We prove that if $S(k) = p$, then $k \leq p!$. Indeed, this follows from the definition (12), since $S(k) = \min\{m \geq 1 : k | m!\} = p$ implies $k | p!$, so $k \leq p!$. Therefore the greatest value of $k$ is $k = p!$, when $S(k) = p | p$. This proves relation (17).

**Theorem 4.** For all primes $p$,

$$S_{\text{min}}(2p) \leq p^2 \leq S_{\text{max}}(2p)$$
and more generally; for all \( m \leq p \),
\[
S_{\min}(mp) \leq p^m \leq S_{\max}(mp)
\] (19)

**Proof.** (19) follows by the known relation \( S(p^m) = mp \) if \( m \leq p \) and the definition (11), (13). Particularly, for \( m = 2 \), (19) reduces to (18). For \( m = p \), (19) gives
\[
S_{\min}(p^2) \leq p^p \leq S_{\max}(p^2)
\] (20)

This case when \( m \) is also an arbitrary prime is given in.

**Theorem 5.** For all odd primes \( p \) and \( q \), \( p < q \) one has
\[
S_{\min}(pq) \leq q^p \leq p^q \leq S_{\max}(pq)
\] (21)
(21) holds also when \( p = 2 \) and \( q \geq 5 \).

**Proof.** Since \( S(q^p) = pq \) and \( S(p^q) = qp \) for primes \( p \) and \( q \), the extreme inequalities of (21) follow from the definition (11) and (13). For the inequality \( q^p < p^q \) remark that this is equivalent to \( f(p) > f(q) \), where \( f(x) = \frac{\ln x}{x} (x \geq 3) \).

Since \( f'(x) = \frac{1-\ln x}{x^2} = 0 \iff x = e \) immediately follows that \( f \) is strictly decreasing for \( x \geq e = 2.71 \). From the graph of this function, since \( \frac{\ln 2}{2} = \frac{\ln 4}{4} \) we get that
\[
\frac{\ln 2}{2} < \frac{\ln 3}{3},
\]
but
\[
\frac{\ln 2}{2} > \frac{\ln q}{q}
\]
for \( q \geq 5 \). Therefore (21) holds when \( p = 2 \) and \( q \geq 5 \). Indeed, \( f(q) \leq f(5) < f(4) = f(2) \).

**Remark.** For all primes \( p, q \)
\[
S_{\min}(pq) \leq \min\{p^q, q^p\}
\] (22)
and
\[
S_{\max}(pq) \geq \max\{p^q, q^p\}.
\] (23)
For \( p = q \) this implies relation (21).

**Proof.** Since \( S(q^p) = S(p^q) = pq \), one has
\[
S_{\min}(pq) \leq p^q, S_{\min}(pq) \leq q^p, S_{\max}(pq) \leq p^q, S_{\max}(pq) \leq q^p
\]
References


