MISCELLANEOUS RESULTS AND THEOREMS ON
SMARANDACHE TERMS AND FACTOR PARTITIONS

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ABSTRACT: In [1] we define SMARANDACHE FACTOR PARTITION FUNCTION (SFP), as follows:

Let \( \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r \) be a set of \( r \) natural numbers and \( p_1, p_2, p_3, \ldots, p_r \) be arbitrarily chosen distinct primes then \( F(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r) \) called the Smarandache Factor Partition of \( (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r) \) is defined as the number of ways in which the number

\[
N = \frac{\alpha_1}{p_1} \frac{\alpha_2}{p_2} \frac{\alpha_3}{p_3} \ldots \frac{\alpha_r}{p_r}
\]

could be expressed as the product of its' divisors. For simplicity, we denote \( F(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r) = F'(N) \), where

\[
N = \frac{\alpha_1}{p_1} \frac{\alpha_2}{p_2} \frac{\alpha_3}{p_3} \ldots \frac{\alpha_r}{p_r} \ldots \frac{\alpha_n}{p_n}
\]

and \( p_r \) is the \( r^{th} \) prime. \( p_1 = 2, p_2 = 3 \) etc.

Also for the case

\[
\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_r = \ldots = \alpha_n = 1
\]

we denote

\[
F(1, 1, 1, 1, 1, \ldots) = F(1#n)
\]

\[
\leftarrow n \text{- ones} \rightarrow
\]

In [2] we define \( b_{(n,r)} x(x-1)(x-2) \ldots(x-r+1)(x-r) \) as the \( r^{th} \) SMARANDACHE TERM in the expansion of
In this note some more results depicting how closely the coefficients of the **Smarandache Term** and SFPs are related are derived.

**DISCUSSION:**

**Result on the \([i^j]\) matrix:**

Theorem (9.1) in [2] gives us the following result

\[ x^n = \sum_{r=0}^{n} x^r a_{(n,r)} \]

which leads us to the following beautiful result.

\[ \sum_{k=1}^{x} k^n = \sum_{k=1}^{x} \sum_{r=1}^{k} k^r a_{(n,r)} \]

In matrix notation the same can be written as follows for \( x = 4 = n \).

\[
\begin{pmatrix}
1P_1 & 0 & 0 & 0 \\
2P_1 & 2P_2 & 0 & 0 \\
3P_1 & 3P_2 & 3P_3 & 0 \\
4P_1 & 4P_2 & 4P_3 & 4P_4
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 3 & 7 \\
0 & 0 & 1 & 6 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1^1 & 1^2 & 1^3 & 1^4 \\
2^1 & 2^2 & 2^3 & 2^4 \\
3^1 & 3^2 & 3^3 & 3^4 \\
4^1 & 4^2 & 4^3 & 4^4
\end{pmatrix}
\]

In general

\[ \mathbf{P} \times \mathbf{A}' = \mathbf{Q} \]

where \( \mathbf{P} = \begin{pmatrix} i^j \end{pmatrix} \)

\( \mathbf{A} = \begin{pmatrix} a_{(i,j)} \end{pmatrix} \) and \( \mathbf{Q} = \begin{pmatrix} i^j \end{pmatrix} \)

(A' is the transpose of A)

Consider the expansion of \( x^n \), again

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\[ x^n = b_{(n,1)} x + b_{(n,2)} x(x-1) + b_{(n,3)} x(x-1)(x-2) + \ldots + b_{(n,n)} x^n \]

for \( x = 3 \) we get

\[ x^3 = b_{(3,1)} x + b_{(3,2)} x(x-1) + b_{(3,3)} x(x-1)(x-2) \]

comparing the coefficient of powers of \( x \) on both sides we get

\[ b_{(3,1)} - b_{(3,2)} + 2 b_{(3,3)} = 0 \]
\[ b_{(3,2)} - 3 b_{(3,3)} = 0 \]
\[ b_{(3,3)} = 1 \]

In matrix form

\[
\begin{bmatrix}
1 & -1 & 2 \\
0 & 1 & -3 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
b_{(3,1)} \\
b_{(3,2)} \\
b_{(3,3)}
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

\[ C_3 \cdot A_3 = B_3 \]
\[ A_3 = C_3^{-1} \cdot B_3 \]

\[
C_3^{-1} = \begin{bmatrix}
1 & -1 & 2 \\
0 & 1 & -3 \\
0 & 0 & 1
\end{bmatrix}^{-1} = \begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 3 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
(C_3^{-1})' = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 3 & 1
\end{bmatrix}
\]
similarly it has been observed that

\[
(C_4^{-1})' = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 3 & 1 & 0 \\
1 & 7 & 6 & 1 \\
\end{bmatrix}
\]

The above observation leads to the following theorem.

**THEOREM (10.1)**

In the expansion of \(x^n\) as

\[
x^n = b_{(n,1)} x + b_{(n,2)} x(x-1) + b_{(n,3)} x(x-1)(x-2) + \ldots + b_{(n,n)} x^n
\]

If \(C_n\) be the coefficient matrix of equations obtained by equating the coefficient of powers of \(x\) on both sides then

\[
(C_n^1)' = \begin{bmatrix}
\alpha_{(i,j)}
\end{bmatrix}_{n \times n} = \text{star matrix of order } n
\]

**PROOF:** It is evident that \(C_{pq}\) the element of the \(p^{th}\) row and \(q^{th}\) column of \(C_n\) is the coefficient of \(x^p\) in \(x^q\). And also \(C_{pq}\) is independent of \(n\). The coefficient of \(x^p\) on the RHS is

coefficient of \(x^p = \sum_{q=1}^{n} b_{(n,q)} C_{pq}\), also

coefficient of \(x^p = 1\) if \(p = n\)

coefficient of \(x^p = 0\) if \(p \neq n\).

in matrix notation

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\[ \text{coefficient of } x^p = \left[ \sum_{q=1}^{n} b_{(n,q)} C_{pq} \right] \]

\[ = \left[ \sum_{q=1}^{n} b_{(n,q)} C'_{qp} \right] \]

\[ = i_{np} \text{ where } i_{np} = 1 \text{, if } n = p \text{ and } i_{np} = 0 \text{, if } n \neq p. \]

\[ = I_n \text{ (identity matrix of order } n). \]

\[ \begin{bmatrix} b_{(n,q)} \end{bmatrix} \begin{bmatrix} C_{p,q} \end{bmatrix}' = I_n \]

\[ \begin{bmatrix} a_{(n,q)} \end{bmatrix} \begin{bmatrix} C_{p,q} \end{bmatrix}' = I_n \text{ as } b_{(n,q)} = a_{(n,q)} \]

\[ A_n \cdot C_n' = I_n \]

\[ A_n = I_n \left[ C_n' \right]^{-1} \]

\[ A_n = \left[ C_n' \right]^{-1} \]

This completes the proof of theorem (10.1).
THEOREM (10.2)

If $C_{k,n}$ is the coefficient of $x^k$ in the expansion of $x^P_n$, then

$$
\sum_{k=1}^{n} F(1#k) C_{k,n} = 1
$$

PROOF: In property (3) of the STAR TRIANGLE following proposition has been established.

$$
F'(1#n) = \sum_{m=1}^{n} a_{(n,m)} = B_n,
$$
in matrix notation the same can be expressed as follows for $n = 4$

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 3 & 7 \\
0 & 0 & 1 & 6 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
= \begin{bmatrix}
B_1 & B_2 & B_3 & B_4
\end{bmatrix}
$$

In general

$$
\begin{bmatrix}
1 \\
\end{bmatrix}_{1 \times n} \ast \begin{bmatrix}
a_{(i,j)} \\
\end{bmatrix}_{n \times n}^{(C_n^{-1})'} = \begin{bmatrix}
B_i \\
\end{bmatrix}_{1 \times n}
$$

$$
\begin{bmatrix}
B_i \\
\end{bmatrix}_{1 \times n} \ast \begin{bmatrix}
\end{bmatrix}_{n \times n}^{(C_n)} = \begin{bmatrix}
1 \\
\end{bmatrix}_{1 \times n}
$$

In $C_{n,n}$, $C_{p,q}$ the $p^{th}$ row and $q^{th}$ column is the coefficient of $x^p$ in $x^P_q$. Hence we have
\[ \sum_{k=1}^{n} F(1\#k) \ C_{k,n} = 1 = \sum_{k=1}^{n} B_{k} \ C_{k,n} \]

**THEOREM (10.3)**

\[ \sum_{k=1}^{n} F(1\#(k+1)) \ C_{k,n} = n + 1 = \sum_{k=1}^{n} B_{k+1} \ C_{k,n} \]

**PROOF:**

It has already been established that

\[ B_{n+1} = \sum_{m=1}^{n} (m+1) \ a_{(n,m)} \]

In matrix notation

\[
\begin{bmatrix}
1 & \cdots & j+1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
a_{(i,j)}
\end{bmatrix}
= 
\begin{bmatrix}
B_{j+1}
\end{bmatrix}
\]

There exist ample scope for more such results.

**REFERENCES:**


[5] "The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texas at Austin, USA.