A functional recurrence to obtain the prime numbers using the Smarandache prime function.

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Theorem: We are considering the function:

For n ≥ 2 , integer:

$$F(n) = n + 1 + \sum_{m=n+1}^{2n} \prod_{i=n+1}^{m} \left[-E\left[-\frac{\sum_{j=1}^{i}\left(E\left(\frac{i}{j}\right)-E\left(\frac{i-1}{j}\right)\right)-2}{\sum_{j=1}^{i}\left(E\left(\frac{i}{j}\right)-E\left(\frac{i-1}{j}\right)\right)-1} \right] \right]$$

one has: p_{k+1} =F(p_k) for all $k \geq 1$ where $\{p_k\}_{k\geq 1}$ are the prime numbers and E(x) is the greatest integer less than or equal to x.

Observe that the knowledge of p_{k+1} only depends on knowledge of p_k and the knowledge of the fore primes is unnecessary.

Observe that this is a functional recurrence strictly closed too.

Proof:

Suppose that we have found a function $G(i)$ with the following property:

$$G(i) = \begin{cases} 1 & \textit{if } i \textit{ is compound} \\ 0 & \textit{if } i \textit{ is prime} \end{cases}$$

This function is called Smarandache Prime Function (Reference)

Consider the following product:

$$\prod_{i=p_k+1}^{m} G(i)$$

If $p_k < m < p_{k+1}$ $\prod_{i=p_k+1}^{m} G(i) = 1$ since $i : p_k + 1 \leq i \leq m$ are all compounds.

If $m \geq p_{k+1}$ $\prod_{i=p_k+1}^{m} G(i) = 0$ since the $G(p_{k+1}) = 0$ factor is in the product.

Here is the sum:

$$\sum_{m=p_k+1}^{2p_k} \prod_{i=p_k+1}^{m} G(i) = \sum_{m=p_k+1}^{p_{k+1}-1} \prod_{i=p_k+1}^{m} G(i) + \sum_{m=p_{k+1}}^{2p_k} \prod_{i=p_k+1}^{m} G(i) = \sum_{m=p_k+1}^{p_{k+1}-1} 1 =$$

$$= p_{k+1} - 1 - (p_k + 1) + 1 = p_{k+1} - p_k - 1$$

The second sum is zero since all products have the factor $G(p_{k+1}) = 0$.

Therefore we have the following relation of recurrence:

$$p_{k+1} = p_k + 1 + \sum_{m=p_k+1}^{2p_k} \prod_{i=p_k+1}^{m} G(i)$$

Let's now see that we can find $G(i)$ with the asked property. Considerer:

$$(1) \quad E\left(\tfrac{i}{j}\right) - E\left(\tfrac{i-1}{j}\right) = \begin{cases} 1 & si\ j \mid i \\ 0 & si\ j \nmid i \end{cases} \qquad j = 1, 2, \ldots, i \qquad i \geq 1$$

We shall deduce this later.

We deduce of this relation:

$$d(i) = \sum_{j=1}^{i} \left(E\left(\tfrac{i}{j}\right) - E\left(\tfrac{i-1}{j}\right) \right)$$ where $d(i)$ is the number of divisors of i.

If i is prime $d(i) = 2$ therefore:

$$-E\left[-\frac{d(i)-2}{d(i)-1}\right] = 0$$

If i is compound $d(i) > 2$ therefore:

$$0 < \frac{d(i)-2}{d(i)-1} < 1 \Rightarrow -E\left[-\frac{d(i)-2}{d(i)-1}\right] = 1$$

Therefore we have obtained the function $G(i)$ which is:

$$G(i) = -E\left[-\frac{\sum_{j=1}^{i}\left(E\left(\frac{i}{j}\right)-E\left(\frac{i-1}{j}\right)\right)-2}{\sum_{j=1}^{i}\left(E\left(\frac{i}{j}\right)-E\left(\frac{i-1}{j}\right)\right)-1} \right] \qquad i \geq 2 \text{ integer}$$

To finish the demonstration of the theorem it is necessary to prove (1)

If $j = 1 \quad j \mid i \quad E\left(\frac{i}{j}\right) - E\left(\frac{i-1}{j}\right) = i - (i-1) = 1$

If $j > 1$

$$i = jE(\tfrac{i}{j}) + r \quad 0 \leq r < j$$
$$i - 1 = jE(\tfrac{i-1}{j}) + s \quad 0 \leq s < j$$

If $j \mid i \Rightarrow r = 0 \Rightarrow jE(\frac{i}{j}) = jE(\frac{i-1}{j}) + s + 1 \Rightarrow \left.\begin{matrix} j \mid s+1 \\ s+1 \leq j \end{matrix}\right\} \Rightarrow j = s+1$

$\Rightarrow jE(\frac{i}{j}) = jE(\frac{i-1}{j}) + j \Rightarrow E(\frac{i}{j}) = E(\frac{i-1}{j}) + 1$

If $j \nmid i \Rightarrow r > 0 \Rightarrow 0 = j(E(\frac{i}{j}) - E(\frac{i-1}{j})) + (r-s) + 1 \Rightarrow j \mid r - s + 1$

Therefore $r - s + 1 = 0$ or $r - s + 1 = j$

If $s \neq 0 \Rightarrow r - s < j - 1 \Rightarrow r - s + 1 = 0 \Rightarrow E(\frac{i}{j}) = E(\frac{i-1}{j})$

If $s = 0 \Rightarrow j \mid i - 1 \Rightarrow E(\frac{i}{j}) = E(\frac{i-1}{j} + \frac{1}{j}) = \frac{i-1}{j} = E(\frac{i-1}{j})$

With this, the theorem is already proved.

Reference:

[1] E. Burton, "Smarandache Prime and Coprime Functions", http://www.gallup.unm.edu/~smarandache/primfnct.txt
[2] F. Smarandache, "Collected Papers", Vol. II, 200 p., p. 137, Kishinev University Press, Kishinev, 1997.

The general term of the prime number sequence and the Smarandache prime function.

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Let 's consider the function $d(i)$ = number of divisors of the positive integer number i. We have found the following expression for this function:

$$d(i) = \sum_{k=1}^{i} E\left(\frac{i}{k}\right) - E\left(\frac{i-1}{k}\right)$$

We proved this expression in the article "A functional recurrence to obtain the prime numbers using the Smarandache Prime Function".

We deduce that the folowing function:

$$G(i) = -E\left[-\frac{d(i)-2}{i}\right]$$

This function is called the Smarandache Prime Function (Reference)
It takes the next values:

$$G(i) = \begin{cases} 0 \text{ if } & i \text{ is prime} \\ 1 \text{ if } i & \text{is compound} \end{cases}$$

Let is consider now $\pi(n)$ =number of prime numbers smaller or equal than n.
It is simple to prove that:

$$\pi(n) = \sum_{i=2}^{n} (1 - G(i))$$

Let is have too:

$$\begin{aligned} &If\ 1 \le k \le p_n - 1 \ \Rightarrow E\left(\frac{\pi(k)}{n}\right) = 0 \\ &If\ \ C_n \ge k \ge p_n \ \ \Rightarrow E\left(\frac{\pi(k)}{n}\right) = 1 \end{aligned}$$

We will see what conditions have to carry C_n.

Therefore we have te following expression for p_n n-th prime number:

$$p_n = 1 + \sum_{k=1}^{C_n} (1 - E\left(\frac{\pi(k)}{n}\right))$$

If we obtain C_n that only depends on n, this expression will be the general term of the prime numbers sequence, since π is in function with G and G does with $d(i)$ that is expressed in function with i too. Therefore the expression only depends on n.

E[x]=The highest integer equal or less than n