On a generalized equation of Smarandache and its integer solutions

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Abstract  Let \( a \neq 0 \) be any given real number. If the variables \( x_1, x_2, \ldots, x_n \) satisfy
\[ x_1 x_2 \cdots x_n = 1, \]
the equation
\[ \frac{1}{x_1} a^{x_1} + \frac{1}{x_2} a^{x_2} + \cdots + \frac{1}{x_n} a^{x_n} = na \]
has one and only one nonnegative real number solution \( x_1 = x_2 = \cdots = x_n = 1 \). This
generalized the problem of Smarandache in book [1].

Keywords  Equation of Smarandache, real number solutions.

§1. Introduction

Let \( Q \) denotes the set of all rational numbers, \( a \in Q \setminus \{−1, 0, 1\} \). In problem 50 of book
[1], Professor F. Smarandache asked us to solve the equation
\[ x a^x + \frac{1}{x} = 2a. \]  \hspace{1cm} (1)
Professor Zhang [2] has proved that the equation has one and only one real number solution
\( x = 1 \). In this paper, we generalize the equation (1) to
\[ \frac{1}{x_1} a^{x_1} + \frac{1}{x_2} a^{x_2} + \cdots + \frac{1}{x_n} a^{x_n} = na, \]  \hspace{1cm} (2)
and use the elementary method and analysis method to prove the following conclusion:

**Theorem.** For any given real number \( a \neq 0 \), if the variables \( x_1, x_2, \ldots, x_n \) satisfy
\( x_1 x_2 \cdots x_n = 1 \), then the equation
\[ \frac{1}{x_1} a^{x_1} + \frac{1}{x_2} a^{x_2} + \cdots + \frac{1}{x_n} a^{x_n} = na \]
has one and only one nonnegative real number solution \( x_1 = x_2 = \cdots = x_n = 1 \).

§2. Proof of the theorem

In this section, we discuss it in two cases \( a > 0 \) and \( a < 0 \).

1) For the case \( a > 0 \), we let
\[ f(x_1, x_2, \ldots, x_{n-1}, x_n) = \frac{1}{x_1} a^{x_1} + \frac{1}{x_2} a^{x_2} + \cdots + \frac{1}{x_{n-1}} a^{x_{n-1}} + \frac{1}{x_n} a^{x_n} - na, \]
If we take \( x_n \) as the function of the variables \( x_1, x_2, \ldots, x_{n-1} \), we have

\[
f(x_1, x_2, \ldots, x_{n-1}, x_n) = \frac{1}{x_1} a^{x_1} + \frac{1}{x_2} a^{x_2} + \cdots + \frac{1}{x_{n-1}} a^{x_{n-1}} + x_1 x_2 \cdots x_{n-1} a^{x_1 x_2 \cdots x_{n-1} - n a}.
\]

Then the partial differential of \( f \) for every \( x_i \) \( (i = 1, 2, \ldots, n - 1) \) is

\[
\frac{\partial f}{\partial x_i} = \frac{1}{x_i} a^{x_i} \left( \log a - \frac{1}{x_i} \right) + \frac{1}{x_i} a^{x_i - 1} \left( x_1 x_2 \cdots x_{n-1} - \log a \right)
\]

\[
= \frac{1}{x_i} \left( a^{x_i} \left( \log a - \frac{1}{x_i} \right) + a^{x_i} \left( \frac{1}{x_n} - \log a \right) \right).
\]

Let

\[
g(x_1, x_2, \ldots, x_{n-1}, x_n) = a^{x_i} \left( \log a - \frac{1}{x_i} \right) + a^{x_n} \left( \frac{1}{x_n} - \log a \right),
\]

the partial differential quotient of \( g \) is

\[
\frac{\partial g}{\partial x_i} = a^{x_i} \left( \log^2 a - \frac{\log a}{x_i} + \frac{a^{x_n}}{x_i} \left( x_1^2 \log^2 a - x_n \log a + 1 \right) \right)
\]

\[
= a^{x_i} \left( \left( x_1 \log a - \frac{1}{2} \right)^2 + \frac{3}{4} \right) + a^{x_n} \left( x_n \log a - \frac{1}{2} \right)^2 + \frac{3}{4} > 0.
\]

It’s easy to prove that the function \( u(x) = a^x (\log a - \frac{1}{x}) \) is increasing for the variable \( x \) when \( x > 0 \). From (3) we have:

i) if \( x_i > x_n, \ g > 0, \ \frac{\partial g}{\partial x_i} > 0 \), and \( f \) is increasing for the variable \( x_i \);

ii) if \( x_i < x_n, \ g < 0, \ \frac{\partial g}{\partial x_i} < 0 \), and \( f \) is decreasing for the variable \( x_i \);

iii) if \( x_i = x_n, \ g = 0, \ \frac{\partial g}{\partial x_i} = 0 \), and we get the minimum value of \( f \).

We have

\[
f \geq f_{x_1=x_n} \geq f_{x_1=x_2=x_n} \geq \cdots \geq f_{x_1=x_2=\cdots=x_n} = 0,
\]

and we prove that the equation (2) has only one integer solution \( x_1 = x_2 = \cdots = x_n = 1 \).

2) For the case \( a < 0 \), the equation (2) can be written as

\[
\frac{1}{x_1} (-1)^{x_1} |a|^{x_1} + \frac{1}{x_2} (-1)^{x_2} |a|^{x_2} + \cdots + \frac{1}{x_n} (-1)^{x_n} |a|^{x_n} = -n|a|,
\]

so we know that \( x_i \ (i = 1, 2, \cdots, n) \) is not an irrational number.

Let \( x_i = \frac{q_i}{p_i} \) (\( q_i \) is coprime to \( p_i \)), then \( p_i \) must be an odd number because negative number has no real square root. From \( x_1 x_2 \cdots x_n = 1 \), we have \( p_1 p_2 \cdots p_n = q_1 q_2 \cdots q_n \), so \( q_i \) is odd number and \( (-1)^{x_i} = -1 \ (i = 1, 2, \cdots, n) \). In this case, the equation (4) become the following equation:

\[
\frac{1}{x_1} |a|^{x_1} + \frac{1}{x_2} |a|^{x_2} + \cdots + \frac{1}{x_n} |a|^{x_n} = n|a|.
\]

From the conclusion of case 1) we know that the theorem is also holds. This completes the proof of the theorem.
References


