ABSTRACT: In this paper, the result (theorem-2.6) Derived in REF. [2], the paper "Generalization Of Partition Function, Introducing ‘Smarandache Factor Partition’ which has been observed to follow a beautiful pattern has been generalized.

DEFINITIONS In [2] we define SMARANDACHE FACTOR PARTITION FUNCTION, as follows:

Let $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r$ be a set of $r$ natural numbers and $p_1, p_2, p_3, \ldots p_r$ be arbitrarily chosen distinct primes then $F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r)$ called the Smarandache Factor Partition of $(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r)$ is defined as the number of ways in which the number

$$N = \frac{\alpha_1 \alpha_2 \alpha_3 \ldots \alpha_r}{p_1 p_2 p_3 \ldots p_r}$$

could be expressed as the product of its’ divisors. For simplicity, we denote $F(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_r) = F'(N)$, where

$$N = \frac{\alpha_1 \alpha_2 \alpha_3 \ldots \alpha_r \alpha_n}{p_1 p_2 p_3 \ldots p_r \ldots p_n}$$

and $p_r$ is the $r^{th}$ prime. $p_1 = 2, p_2 = 3$ etc.

Also for the case
\[ \alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_r = \ldots = \alpha_n = 1 \]

we denote

\[ F(1,1,1,1,1 \ldots) = F(1\#n) \]

\[ \leftarrow n - \text{ones} \rightarrow \]

Smarandache Star Function

\[ (1) \quad F^{**}(N) = \sum_{d|N} F'(d_r) \quad \text{where} \quad d_r | N \]

\[ (2) \quad F^{***}(N) = \sum_{d_r/N} F^{*}(d_r) \]

d_r ranges over all the divisors of \( N \).

If \( N \) is a square free number with \( n \) prime factors, let us denote

\[ F^{***}(N) = F^{**}(1\#n) \]

Here we generalise the above idea by the following definition

Smarandache Generalised Star Function

\[ (3) \quad F^{n*}(N) = \sum_{d_r/N} F^{(n-1)*}(d_r) \quad n > 1 \]

and \( d_r \) ranges over all the divisors of \( N \).

For simplicity we denote

\[ F'(Np_1p_2 \ldots p_n) = F'(N@1\#n) \quad \text{where} \]

\[ (N,p_i) = 1 \quad \text{for} \quad i = 1 \quad \text{to} \quad n \quad \text{and each} \quad p_i \quad \text{is a prime}. \]

\[ F'(N@1\#n) \quad \text{is nothing but the Smarandache factor partition of (a number} \quad N \quad \text{multiplied by} \quad n \quad \text{primes which are coprime to} \quad N). \]
In [3] a proof of the following result is given:

\[ F'(N_{p1}p_{2}p_{3}) = F'^*\(N\) + 3F'^2\(N\) + F'^3\(N\) \]

The present paper aims at generalising the above result.

**DISCUSSION:**

**THEOREM (3.1)**

\[ F'(N@1#n) = F'(N_{p1}p_{2} \ldots p_{n}) = \sum_{m=0}^{n} \left[ a_{(n,m)} F'^{m*}(N) \right] \]

where

\[ a_{(n,m)} = \frac{1}{m!} \sum_{k=1}^{m} (-1)^{m-k} \cdot mC_k \cdot k^n \]

**PROOF:**

Let the divisors of \( N \) be \( d_1, d_2, \ldots, d_k \)

Consider the divisors of \( (N_{p1}p_{2} \ldots p_{n}) \) arranged as follows

\[ d_1, d_2, \ldots, d_k \quad \text{-------say type (0)} \]
\[ d_1p_1, d_2p_1, \ldots, d_kp_1 \quad \text{-------say type (1)} \]
\[ d_1p_1p_j, d_2p_1p_j, \ldots, d_kp_1p_j \quad \text{-------say type (2)} \]
\[ d_1p_1p_j \ldots, d_2p_1p_j \ldots, d_kp_1p_j \ldots \quad \text{-------say type (t)} \]

(there are \( t \) primes in the term \( d_1p_1p_j \ldots \) apart from \( d_1 \))

\[ d_1p_1p_2 \ldots p_n, d_2p_1p_2 \ldots p_n, d_n p_1p_2 \ldots p_n, \quad \text{-------say type (n)} \]

There are \( ^nC_0 \) divisors sets of the type (0)

There are \( ^nC_1 \) divisors sets of the type (1)

There are \( ^nC_2 \) divisors sets of the type (2) and so on

There are \( ^nC_t \) divisors sets of the type (t)
There are \(^nC_n\) divisors sets of the type \((n)\).

Let \(Np_1p_2...p_n = M\). Then

\[
F^*(M) = \sum_{r=0}^{n} \binom{n}{r} \left( \sum \text{of factor partitions of divisors of row } (r) \right)
\]

Let us consider the contributions of divisor sets one by one.

Row (0) or type (0) contributes

\[
F'(d_1) + F'(d_2) + F'(d_3) + \ldots + F'(d_n) = F^*(N)
\]

Row (1) or type (1) contributes

\[
F'(d_1p_1) + F'(d_2p_1) + \ldots F'(d_kp_1)
\]

\[
\sum_{i=1}^{k} F'(d_i) = F^*(d_1) + F^*(d_2) + \ldots + F^*(d_k)
\]

\[
= F^{2*}(N)
\]

Row (2) or type (2) contributes

\[
F'(d_1p_1p_2) + F'(d_2p_1p_2) + \ldots + F'(d_kp_1p_2)
\]

Applying theorem (5) on each of the terms

\[
F'(d_1p_1p_2) = F^*(d_1) + F^{***}(d_1) \quad \text{----(1)}
\]

\[
F'(d_2p_1p_2) = F^*(d_2) + F^{***}(d_2) \quad \text{----(2)}
\]

\[
\vdots
\]

\[
F'(d_kp_1p_2) = F^*(d_k) + F^{***}(d_k) \quad \text{----(k)}
\]

On summing up (1), (2) ... up to (n) we get

\[
F^{2*}(N) + F^{3*}(N)
\]

At this stage let us denote the coefficients as \(a_{(n,r)}\) etc. say 243.
Consider row (t), one divisor set is 
\[ d_1 p_1 p_2 \ldots p_t, d_2 p_1 p_2 \ldots p_t, \ldots d_k p_1 p_2 \ldots p_t, \]
and we have 
\[
F'(d_1 @1\#t) = a_{(t,1)}F^{t*}(d_1) + a_{(t,2)}F^{t2*}(d_1) + \ldots + a_{(t,t)}F^{tt*}(d_1)
\]
\[
F'(d_2 @1\#t) = a_{(t,1)}F^{t*}(d_2) + a_{(t,2)}F^{t2*}(d_2) + \ldots + a_{(t,t)}F^{tt*}(d_2)
\]
\[\vdots\]
\[
F'(d_k @1\#t) = a_{(t,1)}F^{t*}(d_k) + a_{(t,2)}F^{t2*}(d_k) + \ldots + a_{(t,t)}F^{tt*}(d_k)
\]
Summing up both the sides columnwise we get for row (t) or divisors of type (t) one of the \( \binom{n}{t} \) divisor sets contributes 
\[
a_{(t,1)}F^{t2*}(N) + a_{(t,2)}F^{t3*}(N) + \ldots + a_{(t,t)}F^{t(t+1)*}(N)
\]
similarly for row (n) we get 
\[
a_{(n,1)}F^{n2*}(N) + a_{(n,2)}F^{n3*}(N) + \ldots + a_{(n,n)}F^{n(n+1)*}(N)
\]
All the divisor sets of type (0) contribute 
\[
\binom{n}{0} a_{(0,0)}F^{t*}(N) \text{ factor partitions.}
\]
All the divisor sets of type (1) contribute 
\[
\binom{n}{1} a_{(1,1)}F^{t2*}(N) \text{ factor partitions.}
\]
All the divisor sets of type (2) contribute 
\[
\binom{n}{2} \{a_{(2,1)}F^{t2*}(N) + a_{(2,2)}F^{t3*}(N)\} \text{ factor partitions.}
\]
All the divisor sets of type (3) contribute 
\[
\binom{n}{3} \{a_{(3,1)}F^{t2*}(N) + a_{(3,2)}F^{t3*}(N) + a_{(3,3)}F^{t4*}(N)\} \text{ factor partitions.}
\]
All the divisor sets of row (t) or type (t) contribute
\[ ^nC_t \{ a_{(t,1)}F^{1*}(N) + a_{(t,2)}F^{2*}(N) + \ldots + a_{(t,t)}F^{(t+1)*}(N) \} \]
\[ \vphantom{a_{(t,1)}} \]
\[ \vphantom{a_{(t,2)}} \]
All the divisor sets of row (n) or type (n) contribute
\[ ^nC_n \{ a_{(n,1)}F^{1*}(N) + a_{(n,2)}F^{2*}(N) + \ldots + a_{(n,n)}F^{(n+1)*}(N) \} \]
Summing up the contributions from the divisor sets of all the types and considering the coefficient of \( F^{m*}(N) \) for \( m = 1 \) to \( (n+1) \) we get, coefficient of \( F^{1*}(N) \) = \( a_{(0,0)} = 1 = a_{(n+1,1)} \) coefficient of \( F^{12*}(N) \)
\[ = ^nC_1 a_{(1,1)} + ^nC_2 a_{(2,1)} + ^nC_3 a_{(3,1)} + \ldots + ^nC_t a_{(t,1)} + \ldots + ^nC_n a_{(n,1)} \]
\[ = a_{(n+1,2)} \]
coefficient of \( F^{13*}(N) \)
\[ = ^nC_2 a_{(2,2)} + ^nC_3 a_{(3,2)} + ^nC_4 a_{(4,2)} + \ldots + ^nC_t a_{(t,2)} + \ldots + ^nC_n a_{(n,2)} \]
\[ = a_{(n+1,3)} \]
coefficient of \( F^{m*}(N) \)
\[ a_{(n+1,m)} = ^nC_{m-1} a_{(m-1,m-1)} + ^nC_m a_{(m,m-1)} + \ldots + ^nC_n a_{(n,m-1)} \]
coefficient of \( F^{(n+1)*}(N) \)
\[ a_{(n+1,n+1)} = ^nC_n a_{(n,n)} = ^nC_n \cdot ^{n-1}C_{n-1} a_{(n-1,n-1)} = ^nC_n \cdot ^{n-1}C_{n-1} \ldots \]
\[ ^2C_2 \cdot a_{(1,1)} \]
\[ = 1 \]
Consider \( a_{(n+1,2)} \)
\[ = ^nC_1 a_{(1,1)} + ^nC_2 a_{(2,1)} + \ldots + ^nC_t a_{(t,1)} + \ldots + ^nC_n a_{(n,1)} \]

245
\begin{align*}
= \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} \\
= 2^n - 1 \\
= \frac{(2^{n+1} - 2)}{2}.
\end{align*}

Consider \(a_{(n+1,3)}\)

\begin{align*}
= \binom{n}{2} a_{(2,2)} + \binom{n}{3} a_{(3,2)} + \binom{n}{4} a_{(4,2)} + \ldots + \binom{n}{t} a_{(t,2)} + \ldots + \binom{n}{n} a_{(n,2)} \\
= \binom{n}{2}(2^1 - 1) + \binom{n}{3}(2^2 - 1) + \binom{n}{4}(2^3 - 1) + \ldots + \binom{n}{n}(2^{n-1} - 1) \\
= \binom{n}{2}2^1 + \binom{n}{3}2^2 + \ldots + \binom{n}{n}2^{n-1} - \{\binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n}\}
\end{align*}

\begin{align*}
= (1/2) \{ \binom{n}{2}2^2 + \binom{n}{3}2^3 + \ldots + \binom{n}{n}2^n\} - \{ \sum_{r=0}^{n} \binom{n}{r} - \binom{n}{1} - \binom{n}{0} \}  \\
= (1/2) \{ \sum_{r=0}^{n} \binom{n}{r}2^r - \binom{n}{1}.2^1 - \binom{n}{0}.2^0 \} - \{ 2^n - n - 1 \} \\
= (1/2) \{ 3^n - 2n - 1 \} - 2^n + n + 1 \\
= (1/2) \{ 3^n - 2^{n+1} + 1 \} \-----------------(3.1) \\
= (1/3)! \{ (1).3^{n+1} - (3).2^{n+1} + (3).(1)^{n+1} - (1)(0)^{n+1} \}
\end{align*}

Evaluating \(a_{(n+1,4)}\)

\begin{align*}
a_{(n+1,4)} = \binom{n}{3} a_{(3,3)} + \binom{n}{4} a_{(4,3)} + \ldots + \binom{n}{n} a_{(n,3)} \\
= \binom{n}{3}\{3^2 + 1 - 2^3\}/2 + \binom{n}{4}\{3^3 + 1 - 2^4\}/2 + \ldots + \binom{n}{n}\{3^{n-1} + 1 - 2^n\}/2 \\
= (1/2)[\{3^2.\binom{n}{3} + 3^3.\binom{n}{4} + \ldots + 3^{n-1}.\binom{n}{n}\} + \{\binom{n}{3} + \binom{n}{4} + \ldots + \binom{n}{n}\} \\
- \{\binom{n}{3}2^3 + \binom{n}{4}2^4 + \ldots + \binom{n}{n}2^n]\} \\
= (1/2)[(1/3) \{ \sum_{r=0}^{n} \binom{n}{r}3^r - 3^2\binom{n}{2} - 3\binom{n}{1} - \binom{n}{0} \} + \{ \sum_{r=0}^{n} \binom{n}{r} - \binom{n}{2} - \binom{n}{1} \\
- \binom{n}{0} \} - \{ \sum_{r=0}^{n} \binom{n}{r}.2^r - 2^2\binom{n}{2} - 2\binom{n}{1} - \binom{n}{0}\}]
\end{align*}

\begin{align*}
= (1/2) [(1/3)\{4^n - 9n(n-1)/2 - 3n - 1\} + \{2^n - n(n-1)/2 - n - 1\}]
\end{align*}
\[- \{3^n - 4n(n-1)/2 - 2n - 1\}\]

\[a_{(n+1, 4)} = \frac{1}{4!} [ (1) 4^{n+1} - (4) 3^{n+1} + (6) 2^{n+1} - (4) 1^{n+1} + 1(0)^{n+1}]\]

Observing the pattern we can explore the possibility of

\[a_{(n, r)} = \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k \cdot k^n \quad \text{-------(3.2)}\]

which is yet to be established. Now we shall apply induction.

Let the following proposition (3.3) be true for \(r\) and all \(n > r\).

\[a_{(n+1, r)} = \frac{1}{r!} \sum_{k=1}^{r} (-1)^{r-k} \cdot C_k \cdot k^{n+1} \quad \text{-------(3.3)}\]

Given (3.3) our aim is to prove that

\[a_{(n+1, r+1)} = \frac{1}{(r+1)!} \sum_{k=1}^{r+1} [ (-1)^{(r+1)-k} \cdot C_{k+1} \cdot (k+1)^{n+1} ]\]

we have

\[a_{(n+1, r+1)} = ^nC_r \cdot a_{(r, r)} + ^nC_{r+1} \cdot a_{(r+1, r)} + ^nC_{r+2} \cdot a_{(r+2, r)} + \ldots + ^nC_n \cdot a_{(n, r)}\]

\[a_{(n+1, r+1)} = ^nC_r \left[ \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k \cdot k^r \right] + ^nC_{r+1} \left[ \frac{1}{(r+1)!} \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k \cdot k^{r+1} \right]\]

\[+ \ldots + ^nC_n \left[ \frac{1}{(r+1)!} \sum_{k=0}^{r} (-1)^{r-k} \cdot C_k \cdot k^n \right] \]

\[= \frac{1}{r!} \sum_{k=0}^{r} [ (-1)^{r-k} \cdot C_k \{ ^nC_r \cdot k^r + ^nC_{r+1} \cdot k^{r+1} + \ldots + ^nC_n \cdot k^n \}] \]

\[= \frac{1}{r!} \sum_{k=0}^{r} [ (-1)^{r-k} \cdot C_k \{ \sum_{q=0}^{n} C_q \cdot k^q - \sum_{q=0}^{r-1} C_q \cdot k^q \}] \]

\[= \frac{1}{r!} \sum_{k=0}^{r} [ (-1)^{r-k} \cdot C_k \cdot (1+k)^n ] - \frac{1}{(r+1)!} \sum_{q=0}^{r-1} [ (-1)^{r-k} \cdot C_k \{ \sum_{q=0}^{n} C_q \cdot k^q \}]\]
If we denote the 1st and the second term as $T_1$ and $T_2$, we have

$$a_{(n+1,r+1)} = T_1 - T_2 \quad \quad \text{------(3.4)}$$

Consider $T_1 = \frac{1}{r!} \sum_{k=0}^{\infty} [(-1)^{r-k} \cdot \binom{1+r}{k} (1+k)^n]$

$$= \frac{1}{r!} \sum_{k=0}^{\infty} [(-1)^{r-k} \{ r!/(r!(r-k)!) \} (1+k)^n]$$

$$= \frac{1}{(r+1)!} \sum_{k=0}^{\infty} \left[ (-1)^{r-k} \frac{(r+1)!}{(r+1)!(r-k)!} (1+k)^{n+1} \right]$$

$$= \frac{1}{(r+1)!} \sum_{k=0}^{\infty} \left[ (-1)^{r-k} \cdot \binom{r+1}{k} (1+k)^{n+1} \right]$$

$$= \frac{1}{(r+1)!} \sum_{k=0}^{\infty} \left[ (-1)^{r-k} \cdot \binom{r+1}{k+1} (1+k)^{n+1} \right]$$

Let $k + 1 = s$, we get $s = 1$ at $k = 0$ and $s = r + 1$ at $k = r$

$$= \frac{1}{(r+1)!} \sum_{s=1}^{r+1} \left[ (-1)^{(r+1)s} \cdot \binom{r+1}{s} (s)^{n+1} \right]$$

Replacing $s$ by $k$ we get

$$= \frac{1}{(r+1)!} \sum_{k=1}^{r+1} \left[ (-1)^{(r+1)k} \cdot \binom{r+1}{k} (k)^{n+1} \right]$$

In this if we include $k = 0$ case we get

$$T_1 = \frac{1}{(r+1)!} \sum_{k=0}^{r+1} \left[ (-1)^{(r+1)k} \cdot \binom{r+1}{k} (k)^{n+1} \right] \quad \text{------(3.5)}$$

$T_1$ is nothing but the right hand side member of (3.3).

To prove (3.3) we have to prove $a_{(n+1,r+1)} = T_1$

In view of (3.4) our next step is to prove that $T_2 = 0$
\[ T_2 = \frac{1}{r!} \sum_{k=0}^{r} \left( (-1)^{r-k} \cdot C_k \right) \left( \sum_{q=0}^{C_r \cdot k^q} \right) \]

\[ = \frac{1}{r!} \sum_{k=0}^{r} \left( (-1)^{r-k} \cdot C_k \right) \left( \sum_{q=0}^{\binom{n}{k} \cdot k^q} \right) \]

\[ = \frac{1}{r!} \sum_{k=0}^{r} \left( (-1)^{r-k} \cdot C_k \right) + \binom{n}{1} \left( \frac{1}{r!} \sum_{k=0}^{r} \left( (-1)^{r-k} \cdot C_k \right) \frac{k}{r} \right) + \binom{n}{2} \left( \frac{1}{r!} \sum_{k=0}^{r} \left( (-1)^{r-k} \cdot C_k \right) \frac{k^2}{r} \right) + \ldots + \binom{n}{r-1} \left( \frac{1}{r!} \sum_{k=0}^{r} \left( (-1)^{r-k} \cdot C_k \right) \frac{(r-1)!}{r} \right) \]

\[ = \frac{1}{r!} \sum_{k=0}^{r} \left( (-1)^{r-k} \cdot C_k \right) + \binom{n}{1} \left( \frac{1}{r!} \sum_{k=0}^{r} \left( (-1)^{r-k} \cdot C_k \right) \frac{k}{r} \right) + \binom{n}{2} \left( \frac{1}{r!} \sum_{k=0}^{r} \left( (-1)^{r-k} \cdot C_k \right) \frac{k^2}{r} \right) + \ldots + \binom{n}{r-1} \left( \frac{1}{r!} \sum_{k=0}^{r} \left( (-1)^{r-k} \cdot C_k \right) \frac{(r-1)!}{r} \right) \]

\[ = \frac{1}{r!} \sum_{k=0}^{r} \left( (-1)^{r-k} \cdot C_k \right) \]

We shall prove that \( X = 0 \), \( Y = 0 \), \( Z = 0 \) separately.

\[ X = \frac{1}{r!} \sum_{k=0}^{r} \left( (-1)^{r-k} \cdot C_k \right) \]

\[ = \frac{1}{r!} \sum_{k=0}^{r} \left( (-1)^{r-k} \cdot C_k \right) \]

\[ = \frac{1}{r!} \sum_{k=0}^{r} \left( (-1)^{r-k} \cdot C_{r-k} \right) \]

let \( r - k = w \) then we get at \( k = 0 \) \( w = r \) and at \( k = r \) \( w = 0 \).

\[ = \frac{1}{r!} \sum_{w=r}^{0} \left( (-1)^{w} \cdot C_{w} \right) \]

249
\[(1/r!) \sum_{w=0}^{r} \left[ (-1)^w \cdot \binom{C_w}{w} \right] \]

\[= (1 - 1)^r / r! \]

\[= 0 \]

We have proved that \(X = 0\)

(2) \(Y = \binom{n}{1} \sum_{k=0}^{r} \frac{1}{r!} \left\{ (-1)^{r-k} \cdot \binom{C_k}{k} \right\} \)

\[= \binom{n}{1} \sum_{k=1}^{r-1} \frac{1}{(r-1)!} \left\{ (-1)^{r-1-(k-1)} \cdot \binom{C_{k-1}}{k-1} \right\} \]

\[= \binom{n}{1} \frac{1}{(r-1)!} (1 - 1)^{r-1} \]

\[= 0 \]

We have proved that \(Y = 0\)

(3) To prove

\[Z = \binom{n}{2} \cdot a_{(2,r)} + \binom{n}{3} \cdot a_{(3,r)} + \ldots + \binom{n}{r-1} \cdot a_{(r-1,r)} \]

\[= 0 \] ----(3.6)

Proof:

Refer the matrix

\[a_{(1,1)} \quad a_{(1,2)} \quad a_{(1,3)} \quad a_{(1,4)} \ldots \quad a_{(1,r)} \]

\[a_{(2,1)} \quad a_{(2,2)} \quad a_{(2,3)} \quad a_{(2,4)} \ldots \quad a_{(2,r)} \]

\[a_{(3,1)} \quad a_{(3,2)} \quad a_{(3,3)} \quad a_{(3,4)} \ldots \quad a_{(3,r)} \]

\[a_{(4,1)} \quad a_{(4,2)} \quad a_{(4,3)} \quad a_{(4,4)} \ldots \quad a_{(4,r)} \]

\[\ldots\]

250
\[
\begin{array}{cccccc}
\ldots & \ldots & \ldots & \ldots & a_{(r-1,r-1)} & a_{(r-1,r)} \\
a_{(r,1)} & a_{(r,2)} & a_{(r,3)} & \ldots & a_{(r,r-1)} & a_{(r,r)}
\end{array}
\]

The Diagonal elements are underlined. And the the elements above the leading diagonal are shown with bold face.

We have

\[
a_{(1,r)} = \left[ \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} r^k \right] = \frac{y^n}{n!} c_1 = 0 \text{ for } r > 1
\]

All the elements of the first row except \(a_{(1,1)}\) (the one on the leading diagonal) are zero.

Also

\[
a_{(n+1,r)} = a_{(n,r-1)} + r \cdot a_{(n,r)} \quad \text{--------}(3.7)
\]

( This can be easily established by simplifying the right hand side.)

(7) gives us

\[
a_{(2,r)} = a_{(1,r-1)} + r \cdot a_{(1,r)} = 0 \text{ for } r > 2
\]

i.e. \(a_{(2,r)}\) can be expressed as a linear combination of two elements of the first row (except the one on the leading diagonal)

\[
\Rightarrow a_{(2,r)} = 0 \quad r > 2
\]

Similarly \(a_{(3,r)}\) can be expressed as a linear combination of two elements of the second row of the type \(a_{(2,r)}\) with \(r > 3\)

\[
\Rightarrow a_{(2,r)} = 0 \quad r > 3
\]

and so on \(a_{(r-1,r)} = 0\)

substituting

\[
a_{(2,r)} = a_{(3,r)} = \ldots = a_{(r-1,r)} = 0 \text{ in (6)}
\]

we get \(Z = 0\)
With $X = Y = Z = 0$ we get $T_2 = 0$

or $a_{(n+1,r+1)} = T_1 - T_2 = T_1$

from (5) we have

$$T_1 = \frac{1}{(r+1)!} \sum_{k=0}^{r+1} (-1)^{(r+1) - k} \binom{r+1}{k} (k)^{n+1}$$

which gives

$$a_{(n+1,r+1)} = \frac{1}{(r+1)!} \sum_{k=0}^{r+1} (-1)^{(r+1) - k} \binom{r+1}{k} (k)^{n+1}$$

We have proved, if the proposition (3.3) is true for $r$ it is true for ($r+1$) as well. We have already verified it for 1, 2, 3 etc. Hence by induction (3.3) is true for all $n$.

This completes the proof of theorem (3.1).

Remarks: This proof is quite lengthy, clumsy and heavy in algebra. The readers can try some analytic, combinatorial approach.

REFERENCES:


[3] "The Florentine Smarandache " Special Collection, Archives of American Mathematics, Centre for American History, University of Texas at Austin, USA.