HISTORY OF THE SMARANDACHE FUNCTION

I. Bălăcenoiu and V. Seleacu
Department of Mathematics, University of Craiova
Str. Al. I. Cuza 13, Craiova 1100, Romania

1 Introduction

This function is originated from the Romanian professor Florentin Smarandache. It is defined as follows:

For any non-null integer n, $S(n) = \min\{m \mid m! \text{ is divisible by } n\}$.
So we have $S(1) = 0$, $S(2^5) = S(2^6) = S(2^7) = 8$.
If

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \dots \cdot p_t^{\alpha_t} \tag{1}$$

is the decomposition of n into primes, then

$$S(n) = \max S\left(p_i^{\alpha_i}\right) \tag{2}$$

and moreover, if $[m, n]$ is the smallest common multiple of m and n then

$$S([m,n]) = \max\{S(m), S(n)\} \tag{3}$$

Let us observe that if $\wedge = \min$, $\vee = \max$, $\underset{d}{\wedge} =$ the greatest common divizor, $\overset{d}{\vee} =$ the smallest common multiple then S is a function from the lattice $\left(\mathbf{N}^*, \underset{d}{\wedge}, \overset{d}{\vee}\right)$ into the lattice $(\mathbf{N}, \wedge, \vee)$ for which

$$S\left(\overset{d}{\bigvee_{i=\overline{1,s}}} m_i\right) = \bigvee_{i=\overline{1,s}} S(m_i) \tag{4}$$

2 The calculus of S(n)

From (2) it results that to calculate $S(n)$ is necessary and sufficient to know $S\left(p_i^{\alpha_i}\right)$. For this let p be an arbitrary prime number and

$$a_n(p) = \tfrac{p^n-1}{p-1} \qquad b_n(p) = p^n \tag{5}$$

If we consider the usual numerical scale

$$(p):\ b_0(p), b_1(p), \ldots, b_k(p), \ldots$$

and the generalised numerical scale

$$[p]:\ a_1(p), a_2(p), \ldots, a_n(p), \ldots$$

then from the Legendre's formula

$$\alpha! = \prod_{p_i \leq \alpha} p_i^{E_{p_i}(\alpha)} \tag{6}$$

where $E_p(\alpha) = \sum\limits_{j\geq 1}\left[\frac{\alpha}{p^j}\right]$ it results that

$$S\left(p^{a_n(p)}\right) = b_n(p)$$

and even that: if

$$\alpha = k_\nu a_\nu(p) + k_{\nu-1}a_{\nu-1}(p) + \ldots + k_1 a_1(p) = \overline{k_\nu k_{\nu-1}\ldots k_1}_{[p]} \tag{7}$$

is the expression of α in the generalised scale $[p]$ then

$$S\left(p^\alpha\right) = k_\nu p^\nu + k_{\nu-1}p^{\nu-1} + \ldots + k_1 p \tag{8}$$

The right hand in (8) may be written as $p\left(\alpha_{[p]}\right)_{(p)}$. That is $S\left(p^\alpha\right)$ is the number obtained multiplying by p the exponent α written in the scale $[p]$ and "read" it in the scale (p). So, we have

$$S\left(p^\alpha\right) = p\left(\alpha_{[p]}\right)_{(p)} \tag{9}$$

For example to calculate $S\left(3^{100}\right)$ we write the exponent $\alpha = 100$ in the scale

$$[3]:\ 1,\ 4,\ 13,\ 40,\ 121,\ \ldots$$

We have $a_\nu(p) \leq p \Leftrightarrow (p^\nu - 1)/(p-1) \leq \alpha \Leftrightarrow \nu \leq \log_p((p-1)\alpha + 1)$ and so ν is the integer part of $\log_p((p-1)\alpha+1)$,

$$\nu = [\log_p((p-1)\alpha + 1)]$$

For our example $\nu = [\log_3 201] = 4$. Then the first difit of $\alpha_{[3]}$ is $k_4 = [\alpha/a_4(3)] = 2$. So $100 = 2a_4(3) + 20$. For $\alpha_1 = 20$ it results $\nu_1 = [\log_3 41] = 3$ and $k_{\nu_1} = [20/a_3(3)] = 1$ so $20 = a_3(3) + 7$ and we obtain $100_{[3]} = 2a_4(3) + a_3(3) + a_2(3) + 3 = 2113_{[3]}$.
From (8) it results $S(3^{100}) = 3(2113)_{(3)} = 207$.
Indeed, from the Legendre's formula it results that the exponent of the prime p in the decomposition of $\alpha!$ is $\sum\limits_{j\geq 1}\left[\frac{\alpha}{p^j}\right]$, so the exponent of 3 in the decomposition of 207! is $\sum\limits_{j\geq 1}\left[\frac{207}{3^j}\right] = 69 + 23 + 7 + 2 = 101$ and the exponent of 3 in the decomposition of 206! is 99.
Let us observe that, as it is shown in [1], the calculus in the generalised scale $[p]$ is essentially different from the calculus in the standard scale (p), because

$$a_{n+1}(p) = pa_n(p) + 1 \text{ and } b_{n+1}(p) = pb_n(p)$$

Other formulae for the calculus of $S(p^\alpha)$ have been proved in [2] and [3].
If we note $S_p(\alpha) = S(p^\alpha)$ then it results [2] that

$$S_p(\alpha) = (p-1)\alpha + \sigma_{[p]}(\alpha) \tag{10}$$

where $\sigma_{[p]}(\alpha)$ is the sum of the digits of α written in the scale $[p]$

$$\sigma_{[p]}(\alpha) = k_\nu + k_{\nu-1} + \cdots + k_1$$

and also

$$S_p(\alpha) = \frac{(p-1)^2}{p}(E_p(\alpha) + \alpha) + \frac{p-1}{p}\sigma_{(p)}(\alpha) + \sigma_{[p]}(\alpha)$$

where $\sigma_{(p)}(\alpha)$ is the sum of digits of α written in the scale (p), or

$$S_p(\alpha) = p\left(\alpha - \left[\frac{\alpha}{p}\right] + \left[\frac{\sigma_{[p]}(\alpha)}{p}\right]\right)$$

As a direct application of the equalities (2) and (8) in [16] is solved the following problem:

"Which are the numbers with the factorial ending in 1000 zeros ?"

The solution is
$S(10^{1000}) = S(2^{1000}5^{1000}) = \max\{S(2^{1000}), S(5^{1000})\} =$
$= \max\left\{2\left(1000_{[2]}\right)_{(2)}, 5\left(1000_{[5]}\right)_{(5)}\right\} = 4005$. 4005 is the smallest natural number with the asked propriety.
4006, 4007, 4008, and 4009 verify the proprety but 4010 does not, because $4010! = 4009! \cdot 4010$ has 1001 zeros.
In [11] it presents an another calculus formula of $S(n)$:

$$S(n) = n + 1 - \left[\sum_{k=1}^{n} n^{-\left(n\sin\left(k!\frac{\pi}{n}\right)\right)^2}\right]$$

3 Solved and unsolved problems concerning the Smarandache Function

In [16] there are proposed many problems on the Smarandache Function.
M. Mudge in [12] discuses some of these problems. Many of them are unsolved until now. For example:
Problem (i) : Investigate those sets of consecutive integers $i, i+1, i+2, \ldots, i+x$ for which S generates a monotonic increasing (or indeed monotonic decreasing) sequence. (Note: For $1, 2, 3, 4, 5$, S generates the monotonic increasing sequence $0, 2, 3, 4, 5$).
Problem (ii) : Find the smallest integer k for which it is true that for all n less than some given n_0 at least one of $S(n)$, $S(n+1)$, $\ldots$, $S(n-k+1)$ is
(A) a perfect square
(B) a divisor of k^n
(C) a factorial of a positive integer
Conjecture what happens to k as n_0 tends to infinity.
Problem (iii) : Construct prime numbers of the form $\overline{S(n)S(n+1)\ldots S(n+k)}$.
For example $\overline{S(2)S(3)} = 23$ is prime, and $\overline{S(14)S(15)S(16)S(17)} = 75617$ also prime.
The first order forward finite differences of the Smarandache function are defined thus:
$D_s(x) = |S(x+1) - S(x)|$
$D_s^{(k)}(x) = D(D(\ldots \text{ k times } D_s(x)\ldots)$
Problem (iv) : Investigate the conjecture that $D_s^{(k)}(1) = 1$ or 0 for all k

greater than or equal to 2.

J. Duncan in [7] has proved that for the first 32000 natural numbers the conjecture is true.

J. Rodriguez in [14] poses the question than if it is possible to construct an increasing sequence of any (finite) length whose Smarandache values are strictly decreasing. P. Gronas in [9] and K. Khan in [10] give different solution to this question.

T. Yau in [17] ask the question that:

For any triplets of consecutive positive integers, do the values of S satisfy the Fibonacci relationship $S(n)+S(n+1)=S(n+2)$?

Checking the first 1200 positive integers the author founds just two triplets for which this holds:

$S(9)+S(10)=S(11)$, $\quad S(119)+S(120)=S(121)$.

That is $S(11-2)+S(11-1)=S(11)$ and $S(11^2-2)+S(11^2-1)=S(11^2)$ but we observe that $S(11^3-2)+S(11^3-1)\neq S(11^3)$.

More recently Ch. Ashbacher has anounced that for n between 1200 and 1000000 there exists the following triplets satisfying the Fibonacci relationship:

$S(4900)+S(4901)=S(44902)$; $\quad S(26243)+S(26244)=S(26245)$;

$S(32110)+S(32111)=S(32112)$; $\quad S(64008)+S(64009)=S(64010)$;

$S(368138)+S(368139)=S(368140)$; $\quad S(415662)+S(415663)=S(415664)$;

but it is not known if there exists an infinity family of solutions.

The function $C_s\colon \mathbf{N}^*\mapsto \mathbf{Q}$, $C_s(n)=\frac{1}{n}\left(S(1)+S(2)+\cdots+S(n)\right)$ is the sum of Cesaro concerning the function S.

Problem (v) : Is there $\sum\limits_{n\geq 1} C_s^{-1}(n)$ a convergent series? Find the smallest k for which $\left(\underbrace{C_s\circ C_s\circ\cdots\circ C_s}_{k\text{ times}}\right)(m)\geq n$.

Problem (vi) : Study the function $S^{-1}_{\min}\colon \mathbf{N}\setminus\{1\}\mapsto\mathbf{N}$, $S^{-1}_{\min}(n)=\min S^{-1}(n)$, where $S^{-1}(n)=\{m\in\mathbf{N}|S(m)=n\}$.

M. Costewitz in [6] has investigated the problem to find the cardinal of $S^{-1}(n)$.

In [2] it is shown that if for n we consider the standard decomposition (1) and $q_1<q_2<\cdots<q_s<n$ are the primes so that $p_i\neq q_j$, $i=\overline{1,t}$, $j=\overline{1,s}$, then if we note $e_i=E_{p_i}(n)$, $f_k=E_{q_k}(n)$ and $\hat{n}=p_1^{e_1}p_2^{e_2}\cdots p_t^{e_t}$, $\hat{n}_0=\hat{n}/n$,

$q = q_1^{f_1} q_2^{f_2} \cdots q_s^{f_s}$, it result

$$\text{card } S^{-1}(n) = (d(\hat{n}) - d(\hat{n}_0))\, d(q) \tag{11}$$

where $d(r)$ is the number of divisors of r.
The generating function $F_S\colon \mathbf{N}^* \mapsto \mathbf{N}$ associated to S is defined by $F_S(n) = \sum_{d/n} S(d)$. For example $F_S(18) = S(1) + S(2) + S(3) + S(6) + S(9) + S(18) = 20$.
P. Gronas in [8] has proved that the solution of the diophantine equation $F_S(n) = n$ have the solution $n \in \{9, 16, 24\}$ or n prime.
In [11] is investigated the generating function for $n = p^\alpha$. It is shown that

$$F_S(p^\alpha) = (p-1)\frac{\alpha(\alpha+1)}{2} + \sum_{j=1}^{\alpha} \sigma_{[p]}(j) \tag{12}$$

and it is given an algorithm to calculate the sum in the right hand of (12). Also it is proved that $F_S(p_1 p_2 \cdots p_t) = \sum_{i=1}^{t} 2^{i-1} p_i$. Diophantine equations are given in [14] (see also [12]).
We mentione the followings:
(a) $S(x) = S(x+1)$ conjectured to have no solution
(b) $S(mx+n) = x$
(c) $S(mx+n) = m + nx$
(d) $S(mx+n) = x!$
(e) $S(x^m) = x^n$
(f) $S(x) + y = x + S(y)$, x and y not prime
(g) $S(x+y) = S(x) + S(y)$
(h) $S(x+y) = S(x)S(y)$
(i) $S(xy) = S(x)S(y)$
In [1] it is shown that the equation (f) has as solution every pair of composite numbers $x = p(1+q)$, $y = q(1+p)$, where p and q are consecutive primes, and that the equation (i) has no solutions $x, y > 1$.
Smarandache Function Journal, edited at the Department of Mathematics from the University of Craiova, Romania and published by Number Theory Publishing Co, Glendale, Arizona, USA, is a journal devoted to the study of Smarandache function. It publishes original material as well as reprints some that has appeared elsewhere. Manuscripts concerning new results, including computer generated are actively solicited.

4 Generalizations of the Smarandache Function

In [4] are given three generalizations of the Smarandache Function, namely the Smarandache functions of the first kind are the functions $S_n : \mathbf{N}^* \mapsto \mathbf{N}^*$ defined as follows:
(i) if $n = u^i$ ($u = 1$ or $u = p$, prime number) then $S_n(a)$ is the smallest positive integer k with the property that $k!$ is a multiple of n^a .
(ii) if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ then $S_n(a) = \max_{1 \le j \le t} S_{p_j^{\alpha_j}}(a)$.
If $n = p$ then S_n is the function S_p defined by F. Smarandache in [15] ($S_p(a)$ is the smallest positive integer k such that $k!$ is divisible by p^n).
The Smarandache function of the second kind $S^k : \mathbf{N}^* \mapsto \mathbf{N}^*$ are defined by $S^k(n) = S_n(k)$, $k \in \mathbf{N}^*$.
For $k = 1$, the function S^k is the Smarandache function, with the modification that $S(1) = 1$.
If (a): $1 = a_1, a_2, \ldots, a_n, \ldots$
(b): $1 = b_1, b_2, \ldots, b_n, \ldots$
are two sequences with the property that

$$a_{kn} = a_k a_n \quad ; \quad b_{kn} = b_k b_n$$

Let $f_a^b : \mathbf{N}^* \mapsto \mathbf{N}^*$ be the function defined by $f_a^b(n) = S_{a_n}(b_n)$, (S_{a_n} is the Smarandache function of the first kind).
It is easy to see that:
(i) if $a_n = 1$ and $b_n = n$ for every $n \in \mathbf{N}^*$, then $f_a^b = S_1$.
(ii) if $a_n = n$ and $b_n = 1$ for every $n \in \mathbf{N}^*$, then $f_a^b = S^1$.
The Smarandache functions the third kind are functions $S_a^b = f_a^b$ in the case that the sequences (a) and (b) are different from those concerned in the situations (i) and (ii) from above.
In [4] it is proved that

$$S_n(a+b) \le S_n(a) + S_n(b) \le S_n(a) S_n(b) \text{ for } n > 1$$

$$\max\left\{S^k(a), S^k(b)\right\} \le S^k(ab) \le S^k(a) + S^k(b) \text{ for every } a, b \in \mathbf{N}^*$$

$$\max\left\{f_a^b(k), f_a^b(n)\right\} \le f_a^b(kn) \le b_n f_a^b(k) + b_k f_a^b(n)$$

so, for $a_n = b_n = n$ it results

$$\max\left\{S_k(k), S_n(n)\right\} \le S_{kn}(kn) \le n S_k(k) + k S_n(n) \text{ for every } k, n \in \mathbf{N}^* .$$

This relation is equivalent with the following relation written by means of the Smarandache function:

$$\max\left\{S(k^k), S(n^n)\right\} \leq S\left((kn)^{kn}\right) \leq nS(k^k) + kS(n^n)$$

In [5] it is presents an other generalization of the Smarandache function. Let $\mathcal{M} = \{S_m(n) | n, m \in \mathrm{N}^*\}$, let $A, B \in \mathcal{P}(\mathrm{N}^*) \setminus \emptyset$ and $a = \min A$, $b = \min B$, $a^* = \max A$, $b^* = \max B$. The set I is the set of the functions $I_A^B : \mathrm{N}^* \mapsto \mathcal{M}$ with

$$I_A^B(n) = \begin{cases} S_a(b) & , \quad \text{if } n < \max\{a, b\} \\ S_{a_k}(b_k) & , \quad \text{if } \max\{a, b\} \leq n \leq \max\{a^*, b^*\} \\ & \quad\ \text{where} \\ & \quad\ a_k = \max\limits_i \{a_i \in A | a_i \leq n\} \\ & \quad\ b_k = \max\limits_j \{b_j \in B | b_j \leq n\} \\ S_{a^*}(b^*) & , \quad \text{if } n > \max\{a^*, b^*\} \end{cases}$$

Let the rule $\top : I \times I \mapsto I$, $I_A^B \top I_C^D = I_{A\cup C}^{B\cup D}$ and the partial order relation $\rho \subset I \times I$, $I_A^B \rho I_C^D \Leftrightarrow A \subset C$ and $B \subset D$.
It is easy to see that $(I, \top, \rho)$ is a semilattice.
The elements u, $v \in I$ are ρ-strictly preceded by w if:
(i) $w \,\rho\, u$ and $w \,\rho\, v$
(ii) $\forall x \in I \setminus \{w\}$ so that $x \,\rho\, u$ and $x \,\rho\, v \Rightarrow x \,\rho\, w$.
Let $I^\# = \{(u, v) \in I \times I | u, v$ are ρ-strictly preceded $\}$, the rule $\perp : I^\# \mapsto I$, $I_A^B \perp I_C^D = I_{A\cap C}^{B\cap D}$ and the order partial relation r, $I_A^B r I_C^D \Leftrightarrow I_C^D \rho I_A^B$. Then the structure $(I^\#, \perp, r)$ is called the return of semilattice $(I, \top, \rho)$.

References

[1] M. Andrei, C. Dumitrescu, V. Seleacu, L. Tuţescu, Şt. Zanfir : Some remarks on the Smarandache Function (Smarandache Function J. Vol. 4-5, No. 1 (1994), 1-5)

[2] M. Andrei, C. Dumitrescu, V. Seleacu, L. Tuţescu, Şt. Zanfir : La fonction de Smarandache, une nouvelle fonction dans la théorie des nombres (Congrès International H. Poincaré, May 14-18, 1994, Nancy, France)

[3] M. Andrei, I. Bălăcenoiu, C. Dumitrescu, E. Rădescu, N. Rădescu, V. Seleacu : A Linear Combination with the Smarandache Function to obtain the Identity (The 26^{th} Annual Iranian Mathematics Conference, Shahid Bahonar University of Kerman, March 28-31, 1995)

[4] I. Bălăcenoiu : Smarandache Numerical Function (Smarandache Function J. Vol. 4-5, No. 1, (1994), 6-13)

[5] I. Bălăcenoiu : The semilattice with consistent return (Smarandache Notions Journal Vol. 8, 1997)

[6] M. Costewitz : Généralisation du problème 1075 (Smarandache Function J. Vol. 4-5, No. 1, (1994), 43)

[7] J. Duncan : On the conjecture $D_s^{(k)}(1) = 1$ or 0 for $k \geq 2$ (Smarandache Function J. Vol. 2-3, No. 1, (1993), 17-18)

[8] P. Gronas : The solution of the diophantine equation $\sigma_\eta(n) = n$ (Smarandache Function J. Vol. 4-5, No. 1, (1994), 14-16)

[9] P. Gronas : Letter to the Editor (Math. Spectrum, Vol. 27, No. 1, (1994-1995), 21)

[10] K. Khan : Letter to the Editor (Math. Spectrum, Vol. 27, No. 1, (1994-1995), 20-21)

[11] Maohua Le : A formula of the Smarandache Function (Smarandache Notions Journal Vol. 10, (to appear))

[12] M. Mudge : The Smarandache Function toghether with a sample of the Infinity of Unsolved Problems associated with it (Personal Computer World No. 112, (1992), 420)

[13] E. Rădescu, N. Rădescu, C. Dumitrescu : On the Sumatory Function associated to the Smarandache Function (Smarandache Function J. Vol. 4-5, No. 1, (1994), 17-21)

[14] J. Rodriguez : Problem 26.8 (Math. Spectrum, Vol. 26, No. 3, (1994), 91)

[15] F. Smarandache : A function in the Number Theory (An. Univ. Timişoara, Ser. Şt. Mat. Vol. XXVIII, fasc. 1, (1980), 79-88)

[16] F. Smarandache : An Infinity of Unsolved Problems Concerning a Function in the Number Theory (Smarandache Function J. Vol. 1, No. 1, (1990), 12-54)

[17] T. Yau : The Smarandache Function (Math. Spectrum, Vol. 26, No. 3, (1993-1994), 85)