An holomorphic study of Smarandache automorphic and cross inverse property loops

Tèmeítópé Gbóláhán Jaiyéolá

Department of Mathematics, Obafemi Awolowo University, Ile Ife, Nigeria.

Email: jaiyolatemitope@yahoo.com / tjayeola@oauife.edu.ng

Abstract By studying the holomorphic structure of automorphic inverse property quasigroups and loops(AIPQ and (AIPL)) and cross inverse property quasigroups and loops(CIPQ and (CIPL)), it is established that the holomorph of a loop is a Smarandache; AIPL, CIPL, K-loop, Bruck-loop or Kikkawa-loop if and only if its Smarandache automorphism group is trivial and the loop is itself is a Smarandache; AIPL, CIPL, K-loop, Bruck-loop or Kikkawa-loop.

Keywords Smarandache loop, holomorph of loop, automorphic inverse property loop (AIPL), cross inverse property loop(CIPL), K-loop, Bruck-loop, Kikkawa-loop.

§1. Introduction

1.1 Quasigroups and loops
Let $L$ be a non-empty set. Define a binary operation $(\cdot)$ on $L$: If $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid. If the system of equations

$a \cdot x = b$ and $y \cdot a = b$

have unique solutions for $x$ and $y$ respectively, then $(L, \cdot)$ is called a quasigroup. For each $x \in L$, the elements $x^\rho = xJ^\rho, x^\lambda = xJ^\lambda \in L$ such that $xx^\rho = e^\rho$ and $x^\lambda x = e^\lambda$ are called the right, left inverses of $x$ respectively. Now, if there exists a unique element $e \in L$ called the identity element such that for all $x \in L$, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop. To every loop $(L, \cdot)$ with automorphism group $AUM(L, \cdot)$, there corresponds another loop. Let the set $H = (L, \cdot) \times AUM(L, \cdot)$. If we define `$\circ$' on $H$ such that $(\alpha, x) \circ (\beta, y) = (\alpha \beta, x\beta \cdot y)$ for all $(\alpha, x), (\beta, y) \in H$, then $H(L, \cdot) = (H, \circ)$ is a loop as shown in Bruck [7] and is called the Holomorph of $(L, \cdot)$.

A loop(quasigroup) is a weak inverse property loop (quasigroup)[WIPL(WIPQ)] if and only if it obeys the identity $x(yx)^\rho = y^\rho$ or $(xy)^\lambda x = y^\lambda$.

A loop(quasigroup) is a cross inverse property loop(quasigroup)[CIPL(CIPQ)] if and only if it obeys the identity $xy \cdot x^\rho = y$ or $x \cdot yx^\rho = y$ or $x^\lambda \cdot (yx) = y$ or $x^\lambda y \cdot x = y$.

A loop(quasigroup) is an automorphic inverse property loop(quasigroup)[AIPL(AIPQ)] if and only if it obeys the identity $(xy)^\rho = x^\rho y^\rho$ or $(xy)^\lambda = x^\lambda y^\lambda$.
Consider \( (G, \cdot) \) and \( (H, \circ) \) being two distinct groupoids (quasigroups, loops). Let \( A, B \) and \( C \) be three distinct non-equal bijective mappings, that maps \( G \) onto \( H \). The triple \( \alpha = (A, B, C) \) is called an isotopism of \( (G, \cdot) \) onto \( (H, \circ) \) if and only if
\[
xA \circ yB = (x \cdot y)C \forall x, y \in G.
\]

The set \( SYM(G, \cdot) = SYM(G) \) of all bijections in a groupoid \( (G, \cdot) \) forms a group called the permutation (symmetric) group of the groupoid \( (G, \cdot) \). If \( (G, \cdot) = (H, \circ) \), then the triple \( \alpha = (A, B, C) \) of bijections on \( (G, \cdot) \) is called an autotopism of the groupoid (quasigroup, loop) \( (G, \cdot) \). Such triples form a group \( AUT(G, \cdot) \) called the autotopism group of \( (G, \cdot) \). Furthermore, if \( A = B = C \), then \( A \) is called an automorphism of the groupoid (quasigroup, loop) \( (G, \cdot) \). Such bijections form a group \( AUM(G, \cdot) \) called the automorphism group of \( (G, \cdot) \).

The left nucleus of \( L \) denoted by \( N_L(L, \cdot) = \{a \in L : ax \cdot y = a \cdot xy \forall x, y \in L \} \).
The right nucleus of \( L \) denoted by \( N_R(L, \cdot) = \{a \in L : y \cdot xa = yx \cdot a \forall x, y \in L \} \).
The middle nucleus of \( L \) denoted by \( N_M(L, \cdot) = \{a \in L : xa \cdot x = y \cdot ax \forall x, y \in L \} \).
The nucleus of \( L \) denoted by \( N(L, \cdot) = N_L(L, \cdot) \cap N_R(L, \cdot) \cap N_M(L, \cdot) \).
The centrum of \( L \) denoted by \( C(L, \cdot) = \{a \in L : ax = xa \forall x \in L \} \).
The center of \( L \) denoted by \( Z(L, \cdot) = N(L, \cdot) \cap C(L, \cdot) \).

As observed by Osborn [22], a loop is a WIPL and an AIPL if and only if it is a CIPL. The past efforts of Artzy [2], [3], [4] and [5], Belousov and Tzurkan [6] and recent studies of Keedwell [17], Keedwell and Shcherbacov [18], [19] and [20] are of great significance in the study of WIPLs, AIPLs, CIPQs and CIPLs, their generalizations (i.e m-inverse loops and quasigroups, (r,s,t)-inverse quasigroups) and applications to cryptography. For more on loops and their properties, readers should check [8], [10], [12], [13], [27] and [24].

Interestingly, Adeniran [1] and Robinson [25], Oyebola and Adeniran [23], Chiboka and Solarin [11], Bruck [7], Bruck and Paige [9], Robinson [26], Huthnance [14] and Adeniran [1] have respectively studied the holomorphs of Bol loops, central loops, conjugacy closed loops, inverse property loops, A-loops, extra loops, weak inverse property loops, Osborn loops and Bruck loops. Huthnance [14] showed that if \( (L, \cdot) \) is a loop with holomorph \( (H, \circ) \), \( (L, \cdot) \) is a WIPL if and only if \( (H, \circ) \) is a WIPL. The holomorphs of an AIPL and a CIPL are yet to be studied.

For the definitions of inverse property loop (IPL), Bol loop and A-loop readers can check earlier references on loop theory.

Here, a K-loop is an A-loop with the AIP, a Bruck loop is a Bol loop with the AIP and a Kikkawa loop is an A-loop with the IP and AIP.

### 1.2 Smarandache quasigroups and loops

The study of Smarandache loops was initiated by W. B. Vasantha Kandasamy in 2002. In her book [27], she defined a Smarandache loop (S-loop) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. In [16], the present author defined a Smarandache quasigroup (S-quasigroup) to be a quasigroup with at least a non-trivial associative subquasigroup called a Smarandache subsemigroup (S-subsemigroup). Examples of Smarandache quasigroups are given in Mukibodh [21]. In her book, she introduced over 75 Smarandache concepts on loops. In her first paper [28], on the study of Smarandache notions in algebraic structures, she introduced Smarandache : left(right) alternative loops, Bol loops,
Moufang loops, and Bruck loops. But in [15], the present author introduced Smarandache : inverse property loops (IPL), weak inverse property loops (WIPL), G-loops, conjugacy closed loops (CC-loop), central loops, extra loops, A-loops, K-loops, Bruck loops, Kikkawa loops, Burn loops and homogeneous loops.

A loop is called a Smarandache A-loop (SAL) if it has at least a non-trivial subloop that is a A-loop.

A loop is called a Smarandache K-loop (SKL) if it has at least a non-trivial subloop that is a K-loop.

A loop is called a Smarandache Bruck-loop (SBRL) if it has at least a non-trivial subloop that is a Bruck-loop.

A loop is called a Smarandache Kikkawa-loop (SKWL) if it has at least a non-trivial subloop that is a Kikkawa-loop.

If $L$ is a S-groupoid with a S-subsemigroup $H$, then the set $SYM(L, \cdot) = SYM(L)$ of all bijections $A$ in $L$ such that $A : H \to H$ forms a group called the Smarandache permutation (symmetric) group of the S-groupoid. In fact, $SYM(L) \leq SYM(L)$.

The left Smarandache nucleus of $L$ denoted by $SN(L, \cdot) = N_{\lambda}(L, \cdot) \cap H$. The right Smarandache nucleus of $L$ denoted by $SN_{\rho}(L, \cdot) = N_{\rho}(L, \cdot) \cap H$. The middle Smarandache nucleus of $L$ denoted by $SN(L, \cdot) = N(L, \cdot) \cap H$. The Smarandache centrum of $L$ denoted by $SC(L, \cdot) = C(L, \cdot) \cap H$. The Smarandache center of $L$ denoted by $SZ(L, \cdot) = Z(L, \cdot) \cap H$.

**Definition 1.1.** Let $(L, \cdot)$ and $(G, \circ)$ be two distinct groupoids that are isotopic under a triple $(U, V, W)$. Now, if $(L, \cdot)$ and $(G, \circ)$ are S-groupoids with S-subsemigroups $L'$ and $G'$ respectively such that $A : L' \to G'$, where $A \in \{U, V, W\}$, then the isotopism $(U, V, W) : (L, \cdot) \to (G, \circ)$ is called a Smarandache isotopism (S-isotopism).

Thus, if $U = V = W$, then $U$ is called a Smarandache isomorphism, hence we write $(L, \cdot) \cong (G, \circ)$.

But if $(L, \cdot) = (G, \circ)$, then the autotopism $(U, V, W)$ is called a Smarandache autotopism (S-autotopism) and they form a group $SAUT(L, \cdot)$ which will be called the Smarandache autotopism group of $(L, \cdot)$. Observe that $SAUT(L, \cdot) \leq AUT(L, \cdot)$. Furthermore, if $U = V = W$, then $U$ is called a Smarandache automorphism of $(L, \cdot)$. Such Smarandache permutations form a group $SAUM(L, \cdot)$ called the Smarandache automorphism group (SAG) of $(L, \cdot)$.

Let $L$ be a S-quasigroup with a S-subgroup $G$. Now, set $H_S = (G, \circ) \times SAUM(L, \cdot)$. If we define ‘$\circ$’ on $H_S$ such that $(\alpha, x) \circ (\beta, y) = (\alpha \beta, x \beta \cdot y)$ for all $(\alpha, x), (\beta, y) \in H_S$, then $H_S(L, \cdot) = (H_S, \circ)$ is a quasigroup.

If in $L$, $s^\lambda \cdot s^\alpha \in SN(L)$ or $s^\alpha \cdot s^\beta \in SN(L) \forall s \in G$ and $\alpha \in SAUM(L, \cdot)$, $(H_S, \circ)$ is called a Smarandache Nuclear-holomorph of $L$, if $s^\lambda \cdot s^\alpha \in SC(L)$ or $s^\alpha \cdot s^\beta \in SC(L) \forall s \in G$ and $\alpha \in SAUM(L, \cdot)$, $(H_S, \circ)$ is called a Smarandache Centrum-holomorph of $L$ hence a Smarandache Central-holomorph if $s^\lambda \cdot s^\alpha \in SZ(L)$ or $s^\alpha \cdot s^\beta \in SZ(L) \forall s \in G$ and $\alpha \in SAUM(L, \cdot)$.

The aim of the present study is to investigate the holomorphic structure of Smarandache AIPLs and CIPLs (SCIPLs and SAIPLs) and use the results to draw conclusions for Smarandache K-loops (SKLs), Smarandache Bruck-loops (SBRLs) and Smarandache Kikkawa-loops (SKWLs). This is done as follows.
1. The holomorphic structure of AIPQs(AIPLs) and CIPQs(CIPLs) are investigated. Necessary and sufficient conditions for the holomorph of a quasigroup(loop) to be an AIPQ(AIPL) or CIPQ(CIPL) are established. It is shown that if the holomorph of a quasigroup(loop) is a AIPQ(AIPL) or CIPQ(CIPL), then the holomorph is isomorphic to the quasigroup(loop). Hence, the holomorph of a quasigroup(loop) is an AIPQ(AIPL) or CIPQ(CIPL) if and only if its automorphism group is trivial and the quasigroup(loop) is an AIPQ(AIPL) or CIPQ(CIPL). Furthermore, it is discovered that if the holomorph of a quasigroup(loop) is a CIPQ(CIPL), then the quasigroup(loop) is a flexible unipotent CIPQ(flexible CIPL of exponent 2).

2. The holomorph of a loop is shown to be a SAIPL, SCIPL, SKL, SBRL or SKWL respectively if and only its SAG is trivial and the loop is a SAIPL, SCIPL, SKL, SBRL, SKWL respectively.

§2. Main results

**Theorem 2.1.** Let \((L, \cdot)\) be a quasigroup(loop) with holomorph \(H(L)\). \(H(L)\) is an AIPQ(AIPL) if and only if

1. \(AUM(L)\) is an abelian group,
2. \((\beta^{-1}, \alpha, I) \in AUT(L) \forall \alpha, \beta \in AUM(L)\) and
3. \(L\) is a AIPQ(AIPL).

**Proof.** A quasigroup(loop) is an automorphic inverse property loop(AIPL) if and only if it obeys the AIP identity.

Using either of the definitions of an AIPQ(AIPL), it can be shown that \(H(L)\) is a AIPQ(AIPL) if and only if \(AUM(L)\) is an abelian group and \((\beta^{-1}, \alpha, I) \in AUT(L) \forall \alpha, \beta \in AUM(L)\). \(L\) is isomorphic to a subquasigroup(subloop) of \(H(L)\), so \(L\) is a AIPQ(AIPL) which implies \((J_{\rho}, J_{\rho}, J_{\rho}) \in AUT(L)\). So, \((\beta^{-1}, \alpha, I) \in AUT(L) \forall \alpha, \beta \in AUM(L)\).

**Corollary 2.1.** Let \((L, \cdot)\) be a quasigroup(loop) with holomorph \(H(L)\). \(H(L)\) is a CIPQ(CIPL) if and only if

1. \(AUM(L)\) is an abelian group,
2. \((\beta^{-1}, \alpha, I) \in AUT(L) \forall \alpha, \beta \in AUM(L)\) and
3. \(L\) is a CIPQ(CIPL).

**Proof.** A quasigroup(loop) is a CIPQ(CIPL) if and only if it is a WIPQ(WIPL) and an AIPQ(AIPL). \(L\) is a WIPQ(WIPL) if and only if \(H(L)\) is a WIPQ(WIPL).

If \(H(L)\) is a CIPQ(CIPL), then \(H(L)\) is both a WIPQ(WIPL) and a AIPQ(AIPL) which implies 1., 2., and 3. of Theorem 2.1. Hence, \(L\) is a CIPQ(CIPL). The converse follows by just doing the reverse.

**Corollary 2.2.** Let \((L, \cdot)\) be a quasigroup(loop) with holomorph \(H(L)\). If \(H(L)\) is an AIPQ(AIPL) or CIPQ(CIPL), then \(H(L) \cong L\).
Proof. By 2. of Theorem 2.1, \((\beta^{-1}, \alpha, I) \in AUT(L) \forall \alpha, \beta \in AUM(L)\) implies \(x\beta^{-1} \cdot y\alpha = x \cdot y\) which means \(\alpha = \beta = I\) by substituting \(x = e\) and \(y = e\). Thus, \(AUM(L) = \{I\}\) and so \(H(L) \cong L\).

Theorem 2.2. The holomorph of a quasigroup(loop) \(L\) is a AIPQ(AIPL) or CIPQ(CIPL) if and only if \(AUM(L) = \{I\}\) and \(L\) is a AIPQ(AIPL) or CIPQ(CIPL).

Proof. This is established using Theorem 2.1, Corollary 2.1 and Corollary 2.2.

Theorem 2.3. Let \((L, \cdot)\) be a quasigroup(loop) with holomorph \(H(L)\). \(H(L)\) is a CIPQ(CIPL) if and only if \(AUM(L)\) is an abelian group and any of the following is true for all \(x, y \in L\) and \(\alpha, \beta \in AUM(L)\).

1. \((x \beta \cdot y) x^\rho = y^\alpha\).
2. \(x \beta \cdot y x^\rho = y^\alpha\).
3. \((x^\lambda \alpha^{-1} \beta \cdot y) \cdot x = y\).
4. \((x^\lambda \alpha^{-1} \beta \cdot y) \cdot (y^\alpha \cdot x) = y\).

Proof. This is achieved by simply using the four equivalent identities that define a CIPQ(CIPL):

Corollary 2.3. Let \((L, \cdot)\) be a quasigroups(loop) with holomorph \(H(L)\). If \(H(L)\) is a CIPQ(CIPL) then, the following are equivalent to each other

1. \((\beta^{-1} J\rho, \alpha J\rho, J\rho) \in AUT(L) \forall \alpha, \beta \in AUM(L)\).
2. \((\beta^{-1} J\lambda, \alpha J\lambda, J\lambda) \in AUT(L) \forall \alpha, \beta \in AUM(L)\).
3. \((x \beta \cdot y) x^\rho = y^\alpha\).
4. \(x \beta \cdot y x^\rho = y^\alpha\).
5. \((x^\lambda \alpha^{-1} \beta \cdot y) \cdot x = y\).
6. \((x^\lambda \alpha^{-1} \beta \cdot y) \cdot (y^\alpha \cdot x) = y\).

Hence, \((\beta, \alpha, I), (\alpha, \beta, I), (\beta, I, \alpha), (I, \alpha, \beta) \in AUT(L) \forall \alpha, \beta \in AUM(L)\).

Proof. The equivalence of the six conditions follows from Theorem 2.3 and the proof of Theorem 2.1. The last part is simple.

Corollary 2.4. Let \((L, \cdot)\) be a quasigroup(loop) with holomorph \(H(L)\). If \(H(L)\) is a CIPQ(CIPL) then, \(L\) is a flexible unipotent CIPQ(flexible CIPL of exponent 2).

Proof. It is observed that \(J\rho = J\lambda = I\). Hence, the conclusion follows.

Remark. The holomorphic structure of loops such as extra loop, Bol-loop, C-loop, CC-loop and A-loop have been found to be characterized by some special types of automorphisms such as

1. Nuclear automorphism(in the case of Bol-,CC- and extra loops),
2. central automorphism(in the case of central and A-loops).
By Theorem 2.1 and Corollary 2.1, the holomorphic structure of AIPLs and CIPLs is characterized by commutative automorphisms.

Theorem 2.4. The holomorph $H(L)$ of a quasigroup(loop) $L$ is a Smarandache AIPQ(AIPL) or CIPQ(CIPL) if and only if $SAUM(L) = \{I\}$ and $L$ is a Smarandache AIPQ(AIPL) or CIPQ(CIPL).

Proof. Let $L$ be a quasigroup with holomorph $H(L)$. If $H(L)$ is a SAIPQ(SCIPQ), then there exists a S-subquasigroup $H_S(L) \subset H(L)$ such that $H_S(L)$ is a AIPQ(CIPQ). Let $H_S(L) = G \times SAUM(L)$ where $G$ is the S-subquasigroup of $L$. From Theorem 2.2, it can be seen that $H_S(L)$ is a AIPQ(CIPQ) if and only if $SAUM(L) = \{I\}$ and $G$ is a AIPQ(CIPQ). So the conclusion follows.

Corollary 2.5. The holomorph $H(L)$ of a loop $L$ is a SKL or SBRL or SKWL if and only if $SAUM(L) = \{I\}$ and $L$ is a SKL or SBRL or SKWL.

Proof. Let $L$ be a loop with holomorph $H(L)$. Consider the subloop $H_S(L)$ of $H(L)$ such that $H_S(L) = G \times SAUM(L)$ where $G$ is the subloop of $L$.

1. Recall that by [Theorem 5.3, [9]], $H_S(L)$ is an A-loop if and only if it is a Smarandache Central-holomorph of $L$ and $G$ is an A-loop. Combing this fact with Theorem 2.4, it can be concluded that: the holomorph $H(L)$ of a loop $L$ is a SKL if and only if $SAUM(L) = \{I\}$ and $L$ is a SKL.

2. Recall that by [25] and [1], $H_S(L)$ is a Bol loop if and only if it is a Smarandache Nuclear-holomorph of $L$ and $G$ is a Bol-loop. Combing this fact with Theorem 2.4, it can be concluded that: the holomorph $H(L)$ of a loop $L$ is a SBRL if and only if $SAUM(L) = \{I\}$ and $L$ is a SBRL.

3. Following the first reason in 1., and using Theorem 2.4, it can be concluded that: the holomorph $H(L)$ of a loop $L$ is a SKWL if and only if $SAUM(L) = \{I\}$ and $L$ is a SKWL.

References


